On the Relation between Global Properties of Linear Difference and Differential Equations with Polynomial Coefficients, II

Immink, G.K.

Published in:
Journal of Differential Equations

DOI:
10.1006/jdeq.1996.0093

JOURNAL OF DIFFERENTIAL EQUATIONS **128**, 168–205 (1996)
ARTICLE NO. 0093

On the Relation between Global Properties of Linear Difference and Differential Equations with Polynomial Coefficients, II

G. K. Immink

Institute of Econometrics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands

Received December 21, 1994; revised May 24, 1995

This paper is the second one in a series of three articles dealing with applications of the Mellin transformation to the theory of linear differential and difference equations with polynomial coefficients. In the previous part, we studied the case of a differential equation having at most regular singularities at O and ∞ and arbitrary singularities in the rest of the complex plane. This second part is concerned with differential equations having a regular singularity at ∞ and an irregular one at the origin of the complex plane. Using particular types of Mellin (or Pincherle) transforms of appropriate solutions of the differential equation, we construct two fundamental systems of solutions of an associated difference equation. Both fundamental systems admit the same asymptotic representation as the variable tends to ∞, one in the right half plane and the other in the left half plane. We discuss the corresponding connection problem and its relation to the Stokes phenomenon of the differential equation. © 1996 Academic Press, Inc.

Introduction

We consider linear differential operators $D \in \mathbb{C}[t, \partial]$, where

$$\partial := t \frac{d}{dt}$$

and linear difference operators $\Delta \in \mathbb{C}[x, \tau^{-1}]$, where τ denotes the shift operator defined by

$$\tau y(x) = y(x+1).$$

The differential operator

$$D := \sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl} t^h \partial^l \tag{0.1}$$

0022-0396/96 $18.00

is changed to the linear difference operator

$$\Delta := \sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl}(x-h)^l \, \tau^{-h} \tag{0.2}$$

by means of the transformation defined by the substitutions

$$t \to \tau^{-1}, \qquad \partial \to x.$$

Throughout this paper, we consider pairs (D, Δ) of a differential operator D of the type (0.1) and the associated difference operator Δ defined by (0.2). Our main purpose is to study the relations between local and global properties of solutions of the linear differential equation

$$(D) \quad D\varphi = 0$$

and the corresponding difference equation

$$(\Delta) \quad \Delta y = 0.$$

If φ is a solution of the differential equation (D) then the integral

$$\int_\gamma \varphi(t) \, t^{-x-1} \, dt$$

formally satisfies the difference equation (Δ). This type of integral is usually called a Mellin or Pincherle transform of φ, and will play an important role in the present paper. In the study of local properties of both equations the Newton polygons of the operators D and Δ are useful tools. These are defined as follows. The Newton polygon $N(D)$ of a differential operator D of the form (0.1) is the convex hull in $\mathbb{R}^2$ of the set consisting of the half lines γ_{hl}:

$$\gamma_{hl} = \{(x, y) \in \mathbb{R}^2 \colon x \leqslant l, \, y = h\}$$

where $h \in \{0, ..., M\}$ and $l \in \{0, ..., m\}$ such that $a_{hl} \neq 0$. It has two horizontal edges of finite length, a finite number of edges with a positive or negative slope and at most one vertical edge (with slope $= \infty$). If $N(D)$ has an edge with slope k, then $l(k)$ and $h(k)$ will denote its projection on the x-axis and the y-axis, respectively. It is known that

$$\sum_{k>0} h(k) = i_0(D) = \dim H^1(S^1, \mathscr{A}_D^{<0}).$$

Here $i_0(D)$ is the irregularity at O of D, S^1 denotes the unit circle and $\mathscr{A}_D^{<0}$ is the sheaf on S^1 of analytic solutions of (D) that are asymptotically equal to 0 as $t \to 0$ in some sector with vertex at O. If $N(D)$ has an edge with slope $k > 0$, then the equation (D) has $l(k)$ formal solutions of the type

$$\hat{\phi}(t) = e^{p(t)} t^{\rho} \sum_{l=0}^{s} \hat{g}_l(t)(\log t)^l \tag{0.3}$$

where $\rho \in \mathbb{C}$, $p \in \mathbb{C}[t^{-1/q}]$ for some $q \in \mathbb{N}$, with $\deg p = qk$, $s \in \mathbb{N} \cup \{0\}$, $\hat{g}_l \in \mathbb{C}[\![t^{1/q}]\!]$ for each $l \in \{0, ..., s\}$ and the sets of "formal invariants" $\{p, \rho, s\}$ are distinct. It is easily seen that a change of variable $t \mapsto u := (1/t)$ corresponds to a reflection of the Newton polygon in the x-axis. Thus the irregularity $i_\infty(D)$ of D at ∞ equals

$$i_\infty(D) = \sum_{k<0} h(k).$$

Furthermore, $h(\infty)$ equals the total number of non-zero roots of the coefficient of the highest order derivative, i.e., $\sum_{h=0}^{M} a_{hm} t^h$ (provided each root is counted according to its multiplicity).

The Newton polygon $N(\Delta)$ of Δ is obtained from $N(D)$ by a reflection in the line $y = -x$ (cf. [2, 6]). The projection on the horizontal axis of an edge of $N(\Delta)$ with slope d equals the number of formal solutions of (Δ) of the form

$$x^{dx+\rho} e^{\tilde{p}(x)} \sum_{l=0}^{s} \hat{h}_l(x)(\log x)^l \tag{0.4}$$

where $\rho \in \mathbb{C}$, $\tilde{p} \in \mathbb{C}[x^{1/q}]$ for some $q \in \mathbb{N}$, with $\deg \tilde{p} \leqslant q$, $s \in \mathbb{N} \cup \{0\}$, $\hat{h}_l \in \mathbb{C}[\![x^{-1/q}]\!]$ for each $l \in \{0, ..., s\}$ and the sets of "formal invariants" $\{\tilde{p}(\mathrm{mod}\, 2\pi i x\mathbb{Z}), \rho, s\}$ are distinct. A change of the dependent variable of the type

$$y(x) \mapsto z(x) := \Gamma(x)^{\sigma} y(x) \tag{0.5}$$

transforms the difference equation (Δ) into a difference equation (Δ'). If $N(\Delta)$ has an edge with slope d, then the corresponding edge of $N(\Delta')$ will have the slope $d + \sigma$. By choosing a sufficiently large number σ we can see to it that the transformed operator Δ' has a Newton polygon with exclusively positive slopes. The corresponding differential operator D' in that case has a Newton polygon without negative (or infinite) slopes. This implies that $h(\infty) = i_\infty(D') = 0$ and thus D' has an irregular singularity at O, a regular one at ∞ and no other singularities in the finite complex plane. See Fig. 1.

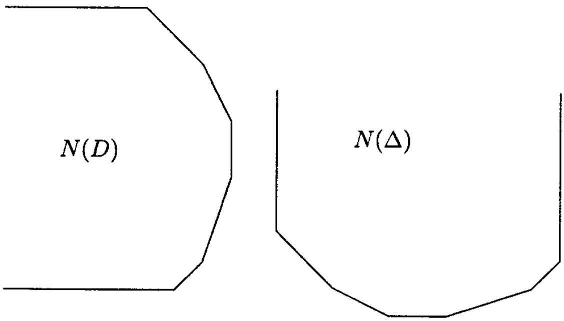

FIG. 1. Example of Newton polygons of D and Δ.

In part I of this paper ([7]) we discussed the particular case that D has at most regular singularities at O and ∞ (and arbitrary singularities in the rest of the complex plane). Under these conditions $N(D)$ has one vertical edge and the difference operator Δ has a Newton polygon with one horizontal edge. We defined fundamental systems of solutions of (Δ) in a right and in a left half plane, using certain types of Mellin transforms of micro-solutions of (D), and expressed the relation between these two systems in terms of central connection matrices of (D). (Note that the operator (Δ) associated with (D) in this paper is, but for a very minor difference, the same as the operator (Δ') defined in [7].)

In this second part we deal with the case that D has at most two singularities: an irregular one at O and a regular one at ∞, i.e., $i_0(D) > 0$ and $h(\infty) = i_\infty(D) = 0$. The Newton polygon of the corresponding difference operator Δ has exclusively positive slopes. The general case can always be reduced to this one by means of a transformation of the type (0.5). The simplest example is provided by the following pair of operators:

$$D = 1 - t\partial, \qquad \Delta = 1 - (x-1)\,\tau^{-1}.$$

Both $N(D)$ and $N(\Delta)$ have one positive slope, equal to 1. The solutions of (D) are multiples of the function $e^{-1/t}$ and the solutions of (Δ) can be written as a product of the gammafunction and some periodic function of period 1. If $\operatorname{Re} x > 0$, we have

$$y(x) := \Gamma(x) = \int_0^\infty e^{-1/t} t^{-x-1}\, dt. \tag{0.6}$$

By application of the saddlepoint method to this integral, one easily derives the well-known asymptotic representation of the gammafunction, valid as $x \to \infty$, $\arg x \in (-\pi/2, \pi/2)$,

$$\hat{y}(x) := x^{x-1/2} e^{-x} \hat{h}(x)$$

where $\hat{h} \in \mathbb{C}[\![x^{-1}]\!]$. Note that the saddle point $t(x) = x^{-1}$ of $e^{-1/t}t^{-x}$ lies in a sector of decrease of $|e^{-1/t}|$ for all x with $\arg x \in (-\pi/2, \pi/2)$. A second solution $\tilde{y}$ of (Δ), which is analytic in a left half plane, can be defined as follows,

$$\tilde{y}(x) := \int_{\gamma} e^{-1/t} t^{-x-1}\, dt \tag{0.7}$$

where γ is a path consisting of the directed segments $(0, \varepsilon)$, $(\varepsilon e^{-2\pi i}, 0)$ and a circular path about O from ε to $\varepsilon e^{-2\pi i}$. Again by means of the saddle point method, it can be shown that $\tilde{y}(x) \sim \hat{y}(x)$ as $x \to \infty$, $\arg x \in \{\pi/2, 3\pi/2\}$. In this case, the saddle point $t(x)$ lies in a sector of increase of $|e^{-1/t}|$. It is useful to keep this example in mind in the study of more general types of operators D and Δ. Like in the example, we shall use integral transforms of solutions of (D) with particular asymptotic properties in order to define solutions of (Δ) admitting corresponding asymptotic representations in a right or left half plane. In general, however, no solution of (D) has the required asymptotic behaviour in a sufficiently large sector. Therefore, we shall have to use several different solutions of (D) with the same asymptotic behaviour in smaller sectors covering the larger one.

It is known that a fundamental system of solutions of (Δ) in a right half plane can be constructed with the aid of analytic Mellin transforms of a basis of $H^1(S^1, \mathscr{A}_D^{<0})$. This idea was put forward by J. P. Ramis in an unpublished preprint ([10]) and used by A. Duval (cf. [5]) and A. Barkatou ([2]). These authors obtained a fundamental system of solutions admitting asymptotic representations of the form (0.4) in a small sector about the positive real axis. In Section 3 of this paper we define a fundamental system Y of solutions of (Δ) that is represented asymptotically by a fundamental system $\hat{Y}$ of formal solutions of (Δ) as $x \to \infty$ in the entire right half plane. In Section 4 we introduce a fundamental system $\tilde{Y}$ admitting the asymptotic representation $\hat{Y}$ in a left half plane. This is the most delicate part of the paper. The derivation of the asymptotic properties of $\tilde{Y}$ is based on a straightforward application of the saddle-point method, but the argument involved is lengthy and technical. The paper is concluded with a discussion of the relation between the fundamental systems Y and $\tilde{Y}$ and its connection with the Stokes phenomenon (at O) of (D). This relation is illustrated with two simple examples.

By a combination of techniques used in this and the previous part of the paper, it is possible to deal with the general case directly, without the need for a preliminary transformation of the type (0.5). This will be explained in Part III ([8]).

1. Preliminaries

We shall use various types of Mellin (or Pincherle) transforms of solutions of the differential equation (D) to represent solutions of (Δ). The following result is well-known.

Proposition 1.1. *Let $\theta \in \mathbb{R}$ and $r > 0$. Let ϕ be a continuous function on the half line γ_θ from O to ∞ with direction θ, admitting an asymptotic representation of the form*

$$\phi(t) \sim \sum_{n=0}^{\infty} \sum_{h=0}^{H} \phi_{nh} t^{\rho_n} (\log t)^h$$

where $\{\rho_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with the property that $\lim_{n\to\infty} \operatorname{Re} \rho_n = \infty$, *as* $t \to 0$, $\arg t = \theta$, *and an asymptotic representation of the form*

$$\phi(t) \sim \sum_{n=0}^{\infty} \sum_{h=0}^{H} \psi_{nh} t^{\sigma_n} (\log t)^h$$

where $\{\sigma_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with the property that $\lim_{n\to\infty} \operatorname{Re} \sigma_n = -\infty$, *as* $t \to \infty$, $\arg t = \theta$. *Then the integral*

$$\int_0^{re^{i\theta}} \phi(t)\, t^{-x-1}\, dt \tag{1.1}$$

defines a meromorphic function with poles at the points ρ_n, where n is a nonnegative integer such that $\phi_{nh} \neq 0$ for some $h \in \{0, ..., H\}$. Moreover, the integral

$$\int_{re^{i\theta}}^{\infty e^{i\theta}} \phi(t)\, t^{-x-1}\, dt \tag{1.2}$$

defines a meromorphic function with poles at the points σ_n, where n is a nonnegative integer such that $\psi_{nh} \neq 0$ for some $h \in \{0, ..., H\}$.

Definition 1.2. Let $\theta \in \mathbb{R}$ and let ϕ be a function satisfying the conditions of Proposition 1.1. By $\mathscr{P}_{\gamma_\theta}(\phi)$ we denote the sum of the meromorphic functions defined by (1.1) and (1.2).

(Again there is a slight difference between the above definition of $\mathscr{P}_{\gamma_\theta}(\phi)$ and the one used in [7].)

Definition 1.3. Let $\Delta \in \mathbb{C}[x, \tau^{-1}]$ be a difference operator of order M. A set of M meromorphic solutions $\{y_1, ..., y_M\}$ of the homogeneous linear difference equation (Δ) is called a *fundamental system* of meromorphic solutions of (Δ) if any meromorphic solution y of (Δ) can be written in the form

$$y = \sum_{i=1}^{M} \pi_i y_i$$

where π_i is a periodic function of period 1, meromorphic in $\mathbb{C}$, for $i = 1, ..., M$.

PROPOSITION 1.4 (cf. [9]). *Let $\Delta \in \mathbb{C}[x, \tau^{-1}]$ be a difference operator of order M. Let $y_1, ..., y_M$ be meromorphic solutions of (Δ) with the property that the so-called "Casorati-determinant" $C_{y_1, ..., y_M}$, defined by*

$$C_{y_1, ..., y_M} = \begin{vmatrix} y_1 & y_2 & \cdot & \cdot & y_M \\ \tau^{-1} y_1 & \tau^{-1} y_2 & \cdot & \cdot & \tau^{-1} y_M \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \tau^{-M+1} y_1 & \tau^{-M+1} y_2 & \cdot & \cdot & \tau^{-M+1} y_M \end{vmatrix}$$

does not vanish identically. Then the set $\{y_1, ..., y_M\}$ is a fundamental system of meromorphic solutions of (Δ).

DEFINITION 1.5. Let $I \subset \mathbb{R}$ be an open interval. By $S(I)$ we denote the sector of the Riemann surface of the logarithm defined by

$$S(I) = \{t\colon \arg t \in I\}.$$

Let $R > 0$. By $S(I, R)$ we denote the set

$$S(I, R) = \{t \in S(I)\colon |t| < R\}.$$

We shall call $S(I)$ a "large" and $S(I, R)$ a "small" sector.

DEFINITION 1.6. By $\mathscr{A}$ we denote the sheaf on $\mathbb{R}$ of holomorphic functions in a small sector with vertex at O and by $\mathscr{A}_\theta$ its stalk at $\theta \in \mathbb{R}$, i.e., the set of germs of functions that are holomorphic on $S((\theta - \varepsilon, \theta + \varepsilon), R)$ for some $\varepsilon > 0$ and $R > 0$. If I is an open subset of $\mathbb{R}$ and $f \in \mathscr{A}(I)$, its germ at $\theta \in \mathbb{R}$ is denoted by f_θ.

Let $k > 0$. By $\mathscr{A}^{\leqslant -k}$, $\mathscr{A}^{< -k}$, $\mathscr{A}^{\leqslant k}$ and $\mathscr{A}^{< k}$ we denote the sheaves on $\mathbb{R}$ of holomorphic functions with at least exponential decrease of order k, with "supra-exponential decrease of order k," with at most exponential growth of order k and with "subexponential growth of order k," respectively, in some sector with vertex at O. More precisely, if $I \subset \mathbb{R}$ is an open interval, $\mathscr{A}^{\leqslant -k}(I)$, $\mathscr{A}^{<-k}(I)$, $\mathscr{A}^{\leqslant k}(I)$ and $\mathscr{A}^{<k}(I)$ are the sets of all functions f with the property that, for any closed interval $I' \subset I$, there exists a positive number R such that f is holomorphic on the sector $S(I', R)$, and

$$\sup_{t \in S(I', R)} |f(t)| \, e^{c\,|t|^{-k}} < \infty$$

for some $c > 0$, for all $c > 0$, for some $c < 0$ and for all $c < 0$, respectively.

By $\mathscr{A}^{\leqslant 0}$ we denote the sheaf on $\mathbb{R}$ of holomorphic functions with moderate growth in some sector with vertex at O. By $\mathscr{A}^{<0}$ we denote the sheaf on $\mathbb{R}$ of holomorphic functions that are asymptotically equal to zero as $t \to 0$ in some sector with vertex at O.

Let $k \geqslant 0$. By $\mathscr{A}_D$, $\mathscr{A}_D^{\leqslant -k}$, $\mathscr{A}_D^{<-k}$, $\mathscr{A}_D^{\leqslant k}$ and $\mathscr{A}_D^{<k}$ we denote the subsheaves of $\mathscr{A}$, $\mathscr{A}^{\leqslant -k}$, $\mathscr{A}^{<-k}$, $\mathscr{A}^{\leqslant k}$ and $\mathscr{A}^{<k}$, respectively, consisting of solutions of (D).

2. The Singularity of (D) at O

Let D be a differential operator of the type (0.1) with the property that $a_{Mm} \neq 0$ and $a_{hm} = 0$ for all $h < M$. This implies that all edges of the Newton polygon of D have finite slopes $\geqslant 0$. Thus D has at most two singularities: an irregular one at O and a regular one at ∞. We shall assume that $a_{0l} \neq 0$ for some $l \in \{0, ..., m\}$. In that case the order of the difference equation (Δ) is equal to M.

At O the differential equation possesses m formal solutions of the following form

$$\hat{\psi}^j(t) = \exp p^j(t)\, t^{\rho^j} \hat{g}^j(t), \qquad j = 1, ..., m,$$

where p^j is a polynomial in $t^{-1/q}$ for some $q \in \mathbb{N}$, of degree $qk^j \in \mathbb{N} \cup \{0\}$, $\rho^j \in \mathbb{C}$, and $\hat{g}^j \in \mathbb{C}[[t^{1/q}]][\log t]$. $\hat{g}^j$ has a non-vanishing constant term. Its degree in $\log t$ will be denoted by s^j. The values of the numbers k^j coincide with the values of the slopes of $N(D)$. Obviously, $\sum_{j=1}^m k^j = M$. Moreover, if, for any pair of distinct indices $i, j \in \{1, ..., m\}$, $p^i \equiv p^j$ and $\rho^i = \rho^j$, then $s^i \neq s^j$. We shall assume that

$$k^1 \leqslant k^2 \leqslant \cdots \leqslant k^m.$$

For all $j \in \{1, ..., m\}$ such that $k^j \neq 0$ we put

$$p^j(t) = -\lambda^j t^{-k^j} + q^j(t),$$

where $\lambda^j \in \mathbb{C}^*$ and either $q^j \equiv 0$ or $\deg q^j < qk^j$.

Definition 2.1. Let $\mathscr{J}$ denote the set of all $j \in \{1, ..., m\}$, such that $k^j > 0$. We assume that a fixed value has been assigned to $\arg \lambda^j$ for each $j \in \mathscr{J}$, for example, $\arg \lambda^j \in [0, 2\pi)$ for each $j \in \mathscr{J}$. We define a bijection $n\colon \mathscr{J} \times \mathbb{Z} \to \mathbb{Z}$ and real numbers α_i, $i \in \mathbb{Z}$, in such a manner that the following conditions hold:

(i) for all $j \in \mathscr{J}$, and $l \in \mathbb{Z}$,

$$\alpha_{n(j,\,l)} = \frac{1}{k^j}(\arg \lambda^j + 2l\pi)$$

(ii) $\alpha_i \leqslant \alpha_{i+1}$ and $\alpha_{i+M} = \alpha_i + 2\pi$ for all $i \in \mathbb{Z}$ (cf. Remark 2.2).

Let $i \in \mathbb{Z}$ and let $j \in \mathscr{J}$ and $l \in \mathbb{Z}$ such that $i = n(j, l)$. We define

$$\hat{\psi}_i := \hat{\psi}^j, \quad p_i := p^j, \quad \lambda_i := \lambda^j e^{2l\pi i}, \quad k_i := k^j, \quad \rho_i := \rho^j, \quad s_i := s^j$$

and

$$\hat{g}_i := \hat{g}^j.$$

Furthermore we define the intervals I_i and $\tilde{I}_i$ by

$$I_i := \left(\alpha_i - \frac{\pi}{2k_i}, \alpha_i + \frac{\pi}{2k_i}\right), \qquad \tilde{I}_i := \left(\alpha_i - \frac{3\pi}{2k_i}, \alpha_i - \frac{\pi}{2k_i}\right)$$

and sectors

$$S_i := S(I_i) \qquad \text{and} \qquad \tilde{S}_i := S(\tilde{I}_i).$$

Remark 2.2. The existence of a bijection from $\mathscr{J} \times \mathbb{Z}$ to $\mathbb{Z}$ with the properties mentioned in Definition 2.1 can be deduced from the following observations. If $\hat{\psi}^j$ is a formal solution of (D), then so is $\hat{\psi}^j(te^{2\pi i})$. This implies that there is a $j' \in \{1, ..., m\}$ such that $k^{j'} = k^j$ and $\arg \lambda^{j'} = \arg \lambda^j + 2k^j\pi \bmod 2\pi$. Consequently, for any $j \in \mathscr{J}$ and any $l \in \mathbb{Z}$, there exists $j' \in \mathscr{J}$ and $l' \in \mathbb{Z}$, such that

$$\frac{1}{k^{j'}}(\arg \lambda^{j'} + 2l'\pi) = \frac{1}{k^j}(\arg \lambda^j + 2l\pi) + 2\pi.$$

Furthermore, it is easily verified that, for any given $k > 0$ and any real number α, there are $h(k)$ pairs (j, l) such that $k^j = k$ and $1/k^j(\arg \lambda^j + 2l\pi) \in [\alpha, \alpha + 2\pi)$.

Note that $\operatorname{Re} \lambda_i t^{-k_i}$ increases as $t \to 0$, in S_i and decreases as $t \to 0$ in $\tilde{S}_i$. A direction $\alpha \in \mathbb{R}$ is a *Stokes direction* for (D) if there exist $i, j \in \{1, ..., m\}$ such that $p^i - p^j = \lambda t^{-k} + o(t^{-k})$ as $t \to 0$, where $\lambda \neq 0$, $k > 0$, and

$\alpha = 1/k(\arg \lambda + \pi/2) \bmod \pi/k$. If $k^1 = 0$, the direction $\alpha_i + (2l+1)(\pi/2k_i)$ is a Stokes direction for every i and $l \in \mathbb{Z}$.

Let $\theta \in \mathbb{R}$. It is well-known that, for every $j \in \{1, ..., m\}$, there exists a solution $\psi^{j\theta}$ of (D) which is represented asymptotically by the formal solution $\hat{\psi}^j$ as $t \to 0$, $\arg t \in (\theta - (\pi/2k^m), \theta + (\pi/2k^m))$, uniformly on closed subintervals. The functions $\psi^{1\theta}, ..., \psi^{m\theta}$ constitute a fundamental system of solutions of (D). For all $j \in \mathscr{J}$ there exists exactly one integer i such that $p^j = p_i$ and $\theta \in (\alpha_i - (3\pi/2k_i), \alpha_i + (\pi/2k_i)]$. Let $\psi_i^\theta := (\psi^{j\theta})_\theta$. Then $\{\psi_i^\theta \colon I_i \ni \theta\}$ is a basis of $(\mathscr{A}_D^{\leqslant 0})_\theta$ and $\{\psi_i^\theta (\bmod\, (\mathscr{A}_D^{\leqslant 0})_\theta) \colon [\alpha_i - (3\pi/2k_i), \alpha_i - (\pi/2k_i)] \ni \theta\}$ is a basis of $(\mathscr{A}_D/\mathscr{A}_D^{\leqslant 0})_\theta$.

Definition 2.3. Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. A collection of open intervals $\{I'_\nu = (\alpha'_\nu, \beta_\nu)\}_{\nu=1}^N$, will be called a *good covering* of (α, β) if $\alpha'_1 = \alpha$, $\beta_N = \beta$, $\alpha'_\nu < \alpha'_{\nu+1}$ and $\beta_\nu < \beta_{\nu+1}$ for all $\nu \in \{1, ..., N-1\}$, and, for $\nu \in \{1, ..., N-2\}$,

$$\alpha'_{\nu+1} < \beta_\nu < \alpha'_{\nu+2} < \beta_{\nu+1}.$$

The collection $\{I'_\nu\}_{\nu=1}^N$ will be called a good covering of $[\alpha, \beta)$, of $(\alpha, \beta]$ and of $[\alpha, \beta]$, respectively, if $\alpha'_1 < \alpha$ and $\beta_N = \beta$, $\alpha'_1 = \alpha$ and $\beta_N > \beta$, and $\alpha'_1 < \alpha$ and $\beta_N > \beta$, respectively.

In the next two sections we shall construct solutions of (Δ) admitting certain asymptotic representations $\hat{y}_i$ $(i \in \mathbb{Z})$. These asymptotic representations will be derived from the formal solutions $\hat{\psi}_i$ of (D) defined above, by application of the saddle point method to integrals of the form

$$\int_\gamma \psi_i(t)\, t^{-x-1}\, dt$$

where ψ_i is a solution of (D) with the property that $\psi_i(t) \sim \hat{\psi}_i(t) = e^{p_i(t)} t^{\rho_i} \hat{g}_i(t)$ as $t \to 0$ in some sector of the Riemann surface of $\log t$. The function $e^{p_i(t)} t^{-x}$ has a saddle point $t_i(x)$ which, in first approximation, is equal to $(x/k_i\lambda_i)^{-1/k_i}$ and thus $\arg t_i(x) \simeq \alpha_i - (1/k_i) \arg x$. Therefore, in order to obtain solutions of (Δ) which are represented asymptotically by $\hat{y}_i$ as $x \to \infty$, $\arg x \in (-\pi/2, \pi/2)$, or $\arg x \in (\pi/2, 3\pi/2)$, we need suitable collections $\{\psi_{i,\nu}\}$ of solutions of (D), represented asymptotically by $\hat{\psi}_i$ as $t \to 0$, $\arg t \in I_{i,\nu}$, where the intervals $I_{i,\nu}$ cover I_i or $\tilde{I}_i$, respectively. The existence of such collections of solutions is derived from the following crucial lemma, which can be deduced from Theorem 6.1 in [1] (cf. also [3]).

Lemma 2.4. *Let $i \in \mathbb{Z}$, $\alpha \in \mathbb{R}$. There exists a positive integer N_i, open intervals $I_{i,\nu} = (\alpha_{i,\nu}, \beta_{i,\nu})$, and solutions $\psi_{i,\nu}$ of (D), $\nu = 0, ..., N_i + 1$, such that the following conditions are satisfied:*

(i) $\{I_{i,\nu}\}_{\nu=0}^{N_i+1}$ *is a good covering of* $[\alpha, \alpha+(\pi/k_i)]$ *and* $\{I_{i,\nu}\}_{\nu=1}^{N_i}$ *is a good covering of* $(\alpha, \alpha+(\pi/k_i))$.

(ii) $\psi_{i,\nu}(t) \sim \hat{\psi}_i(t)$ *as* $t \to 0$, $\arg t \in I_{i,\nu}$, *for all* $\nu \in \{0, ..., N_i+1\}$.

(iii) $\psi_{i,\nu} - \psi_{i,\nu+1} \in \mathscr{A}_D^{<-k_i}(I_{i,\nu} \cap I_{i,\nu+1})$ *for all* $\nu \in \{1, ..., N_i - 1\}$.

(iv) $e^{-p_i(t)}(\psi_{i,\nu} - \psi_{i,\nu+1}) \in \mathscr{A}^{\leqslant -k_i}(I_{i,\nu} \cap I_{i,\nu+1})$ *for* $\nu = 0$ *and* $\nu = N_i$.

Remark 2.5. If $k^1 = k^m = k$ and $\lambda^i \neq \lambda^j$ for all i, j such that $p^i \neq p^j$, then the formal solutions of (D) are k-summable in all directions, except for a finite number of so-called singular directions. In that case we can take $N_i = 1$ and $\psi_{i,1} = e^{p_i(t)} t^{\rho_i} g_{i,1}$, where $g_{i,1}$ is the k-sum of $\hat{g}_i$ in the direction $\alpha + (\pi/2k)$, or, if this happens to be a singular direction, in a direction slightly smaller or larger than $\alpha + (\pi/2k)$. If there are $i, j \in \{1, ..., m\}$, such that $k^i = k^j$, $\lambda^i = \lambda^j$, but $p^i \neq p^j$, or if $k^1 < k^m$, then, in general, the equation (D) has more than one "level" and its formal solutions are not k-summable for any given k, but multi-summable. In that case the $\psi_{i,\nu}$ can be obtained by taking multi-sums of $\hat{g}_i$ in appropriate "multi-directions" $\theta_{i,\nu}$. With a slight abuse of terminology, we shall call such a $\psi_{i,\nu}$ a "multi-sum" of $\hat{\psi}_i$. If the differential equation has r levels: $\kappa_1 < \cdots < \kappa_r$, then the multi-direction $\theta_{i,\nu}$ has r components $\theta_{i,\nu}^1, ..., \theta_{i,\nu}^r$ and the process of multi-summation consists of r steps. Roughly speaking, in the jth step the formal series $\hat{g}_i$ is summed in the direction $\theta_{i,\nu}^j$, modulo functions with exponential decrease of order $> \kappa_j$. By choosing the components of $\theta_{i,\nu}$ corresponding to levels $\leqslant k_i$ approximately equal to $\alpha + (\pi/2k_i)$ for all $\nu \in \{0, ..., N_i\}$, we can arrange for the differences $\psi_{i,\nu} - \psi_{i,\nu+1}$ to contain only contributions from levels $> k_i$. By a judicious choice of the remaining components we can see to it that these differences decrease of exponential order $> k_i$ in the intersections $I_{i,\nu} \cap I_{i,\nu+1}$.

A collection of functions $\{\psi_{i,\nu}\}_{\nu=1}^{N_i}$, defined on a good covering $\{I_{i,\nu}\}_{\nu=1}^{N_i}$ of $(\alpha, \alpha+(\pi/k_i))$, with the properties (ii) and (iii) mentioned in Lemma 2.4, defines a section on $(\alpha, \alpha+(\pi/k_i))$ of the quotient sheaf $\mathscr{A}_D^{\leqslant k_i}/\mathscr{A}_D^{<-k_i}$. In [3] we prove a stronger statement, viz. the existence of a collection of functions $\{\psi_{i,\nu}\}_{\nu=1}^{N_i}$ representing a section on $[\alpha, \alpha+(\pi/k_i))$ or $(\alpha, \alpha+(\pi/k_i)]$ of $\mathscr{A}_D^{\leqslant k_i}/\mathscr{A}_D^{<-k_i}$.

From Remark 2.2 and Lemma 2.4 one easily deduces the following result, which will be used in [8].

PROPOSITION 2.6. *Let* $\phi \in \mathscr{A}_D(\mathbb{R})$. *There is a finite number of directions* $\theta_i \in \mathbb{R}$ *and functions* $\psi_i \in \mathscr{A}_D(\mathbb{R})$, $i \in 1, ..., N$, *such that* $\psi_{i\theta_i} \in (\mathscr{A}_D^{\leqslant 0})_{\theta_i}$ *and* $\phi = \sum_{i=1}^N \psi_i$.

Proof. Choose $\theta \in \mathbb{R}$ such that $\theta \neq \alpha_i - (\pi/2k_i) \bmod \pi/k_i$ for all $i \in \mathbb{Z}$ and let J denote the set of $i \in \mathbb{Z}$ with the property that $\tilde{I}_i \ni \theta$. Let $\theta' < \theta$ such that

$\theta' \in \tilde{I}_i$ for all $i \in J$ and let $\{\psi_{i,\nu}\}_{\nu=1}^{N_i}$ be a collection of solutions of (D), defined on a good covering $\{I_{i,\nu}\}_{\nu=1}^{N_i}$ of $(\theta', \theta' + (\pi/k_i))$, with the properties (ii) and (iii) mentioned in Lemma 2.4. Without loss of generality we may assume that $\theta \in I_{i,1}$ for all $i \in J$. The equivalence classes modulo $(\mathscr{A}_D^{\leqslant 0})_\theta$ of the germs $(\psi_{i,1})_\theta$, $i \in J$, form a basis of $(\mathscr{A}_D/\mathscr{A}_D^{\leqslant 0})_\theta$. Hence there exist complex numbers c_i, $i \in J$, such that

$$\phi_\theta = \tilde{\psi} + \sum_{i \in J} c_i (\psi_{i,1})_\theta$$

where $\tilde{\psi} \in (\mathscr{A}_D^{\leqslant 0})_\theta$. Furthermore,

$$\psi_{i,1} = \psi_{i,N_i} + \sum_{\nu=1}^{N_i - 1} (\psi_{i,\nu} - \psi_{i,\nu+1}).$$

Obviously, $I_{i,N_i} \cap I_i \neq \varnothing$ and $\psi_{i,N_i} \in \mathscr{A}_D^{<0}(I_{i,N_i} \cap I_i)$. Hence the result follows. ∎

Remark 2.7. The discussion in this section only concerns the *local properties* of D at the irregular singular point O. It therefore also applies to the general case of a differential operator with an irregular singularity at O (or, mutatis mutandis, at any point of the extended complex plane), provided the number M is replaced by the irregularity $i_0(D)$ of D at O. The same is true of Lemma 4.1 and Remark 4.2 (cf. also Remark 4.4) in Section 4.

3. The Fundamental System Y

Definition 3.1. Let $i \in \mathbb{Z}$ and suppose we are given intervals $I_{i,\nu}$ and solutions $\psi_{i,\nu}$ of (D), $\nu = 1, ..., N_i$, such that the conditions (i)–(iii) of Lemma 2.4 are satisfied, with $\alpha = \alpha_i - (\pi/2k_i)$. Let $\gamma_{i,\nu}$ denote a half line from O to ∞ in $S(I_{i,\nu})$. We define

$$y_{i,\nu} := \mathscr{P}_{\gamma_{i,\nu}}(\psi_{i,\nu}), \qquad \nu \in \{1, ..., N_i\}.$$

By means of partial integration it can be verified that, for every integer i and every $\nu \in \{1, ..., N_i\}$, $y_{i,\nu}$ is a solution of the difference equation (Δ). According to Proposition 1.1, it is analytic in a right half plane and meromorphic in $\mathbb{C}$. Moreover, it has particular asymptotic properties, as stated in the next theorem. By $t_i(x)$ we denote the unique saddle point of the function $t^{-x} \exp p_i(t)$ belonging to the sector $\tilde{S}_i \cup S_i$. It is easily verified that

$$t_i(x) = \left(\frac{x}{k_i \lambda_i}\right)^{-1/k_i} (1 + o(1))$$

as $x \to \infty$, $\arg x \in (-\pi/2, 3\pi/2)$. Moreover, at ∞, $t_i(x)$ admits a convergent power series expansion in x^{-1/k_i} (cf. [4]).

THEOREM 3.2. *Let* $i \in \mathbb{Z}$, $v \in \{1, ..., N_i\}$. $y_{i,v}$ *admits an asymptotic representation* $\hat{y}_i$ *as* $x \to \infty$, $\arg x \in (-\pi/2, \pi/2)$, *uniformly on closed subintervals. This representation has the following form*

$$\hat{y}_i(x) = x^{-1/2} t_i(x)^{-x+\rho_i} \exp\{p_i(t_i(x))\} (\log x)^{s_i} \hat{f}_i(x) \tag{3.1}$$

where $\tilde{q} \in \mathbb{N}$, $\hat{f}_i \in \mathbb{C}[\![x^{-1/\tilde{q}}]\!][1/\log x]$ *such that* $\hat{f}_i(\infty) \neq 0$, *or, equivalently,*

$$\hat{y}_i(x) = x^{(x-\rho_i)/k_i - 1/2} \exp\{\tilde{p}_i(x)\} (\log x)^{s_i} \hat{h}_i(x), \tag{3.2}$$

where $\tilde{p}_i$ *is a polynomial in* $x^{1/\tilde{q}}$ *of degree not exceeding* $\tilde{q}$ *and* $\hat{h}_i \in \mathbb{C}[\![x^{-1/\tilde{q}}]\!][1/\log x]$, $\hat{h}_i(\infty) \neq 0$.

Proof. By application of the saddle-point method to a suitable integral representation of $y_{i,v}$ it can be shown that

$$y_{i,v}(x) \sim \hat{y}_i(x) \qquad \text{as} \quad x \to \infty, \qquad \arg x \in (k_i(\alpha_i - \beta_{i,v}), k_i(\alpha_i - \alpha_{i,v})) \tag{3.3}$$

(cf. [4, 10]). We shall prove the theorem by means of induction on $k^m - k_i$. Suppose that $k_i = k^m$. As $\mathscr{A}_D^{<-k^m} = \{0\}$, the conditions of Lemma 2.4 imply that $\psi_{i,v} = \psi_{i,v+1}$ for all $v \in \{1, ..., N_i - 1\}$, hence $\psi_{i,v}(t) \sim \hat{\psi}_i(t)$ as $t \to 0$, $\arg t \in I_i$, in this case. Thus we may take $\alpha_{i,v} = \alpha_{i,1} = \alpha_i - (\pi/2k_i)$ and $\beta_{i,v} = \beta_{i,N_i} = \alpha_i + (\pi/2k_i)$ and the result follows immediately from (3.3). Now suppose that $k_i < k^m$ and that the theorem is true for all integers j such that $k_j > k_i$. Let $\theta_{i,v} \in (\alpha_{i,v+1}, \beta_{i,v})$ and let γ_i^v denote the half line from O to ∞ with direction $\theta_{i,v}$. Obviously,

$$y_{i,v} - y_{i,v+1} = \mathscr{P}_{\gamma^v_i}(\psi_{i,v} - \psi_{i,v+1}), \qquad v \in \{1, ..., N_i - 1\}. \tag{3.4}$$

Let $J := \{j \in \mathbb{Z} : k_j > k_i, \theta_{i,v} \in I_j\}$. For every $j \in J$ there is an integer v_{ij} such that $\theta_{i,v} \in I_{j,v_{ij}}$. Furthermore, due to condition (iii) of Lemma 2.4, there exist complex numbers c_j such that

$$\psi_{i,v} - \psi_{i,v+1} = \sum_{j \in J} c_j \psi_{j,v_{ij}}. \tag{3.5}$$

From (3.4) and (3.5) we deduce that

$$y_{i,v} - y_{i,v+1} = \sum_{j \in J} c_j \mathscr{P}_{\gamma^v_i}(\psi_{j,v_{ij}}) = \sum_{j \in J} c_j y_{j,v_{ij}}.$$

According to the induction hypothesis, $y_{j,\,\nu_{ij}}(x) \sim \hat{y}_j(x)$ as $x \to \infty$, $\arg x \in (-\pi/2, \pi/2)$, for all $j \in J$. Hence it follows that, for any $\nu \in \{1, ..., N_i - 1\}$,

$$x^{-x/k_i} \exp(-\tilde{p}_i(x))(y_{i,\,\nu}(x) - y_{i,\,\nu+1}(x)) \sim 0$$

$$\text{as} \quad x \to \infty, \quad \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Consequently, $y_{i,\,\nu}(x) \sim \hat{y}_i(x)$ as $x \to \infty$, $\arg x \in \bigcup_{\mu=1}^{N_i} (k_i(\alpha_i - \beta_{i,\,\mu}), k_i(\alpha_i - \alpha_{i,\,\mu})) = (k_i(\alpha_i - \beta_{i,\,N_i}), k_i(\alpha_i - \alpha_{i,\,1})) = (-\pi/2, \pi/2)$. ∎

For every $i \in \mathbb{Z}$, let us fix an integer $\nu_i \in \{1, ..., N_i\}$ and put $\psi_{i,\,\nu_i} = \psi_i$ and $y_{i,\,\nu_i} = y_i$. For each $i \in \mathbb{Z}$, $\hat{y}_i$ is a formal solution of the difference equation (Δ). The Casorati-determinant $C_{\hat{y}_1, ..., \hat{y}_M}$ of the set of formal solutions $\{\hat{y}_1, ..., \hat{y}_M\}$ has the form

$$C_{\hat{y}_1, ..., \hat{y}_M}(x) = x^{dx+\rho} \lambda^x \sum_{n=0}^{\infty} c_n x^{-n} \tag{3.6}$$

where $d = \sum_{i=1}^{M} (1/k_i)$, $\lambda = \prod_{i=1}^{M} (k_i \lambda_i / e)^{1/k_i}$, $\rho \in \mathbb{C}$ and $c_0 \neq 0$. The Casorati-determinant of the set of meromorphic solutions

$$Y := \{y_i, i \in \{1, ..., M\}\} \tag{3.7}$$

admits the asymptotic representation (3.6) as $x \to \infty$, $\arg x \in (-\pi/2, \pi/2)$ and thus cannot vanish identically. Hence, by Proposition 1.4, this set is a *fundamental system of solutions of* (Δ).

4. The Fundamental System $\tilde{Y}$

We are going to introduce a second type of solutions $\tilde{y}_i$ $(i \in \mathbb{Z})$, which are represented asymptotically by $\hat{y}_i$ as $x \to \infty$ in a left half plane. For this purpose, we need a covering of $[\alpha_i - (3\pi/2k_i), \alpha_i - (\pi/2k_i)]$ with open intervals $I_{i,\,\nu}$ and a collection of solutions $\psi_{i,\,\nu}$ of (D) with the property that $\psi_{i,\,\nu}(t) \sim \hat{\psi}_i(t)$ as $t \to 0$, $\arg t \in I_{i,\,\nu} \cap \tilde{I}_i$. Instead of defining $\tilde{y}_i$ by one contour integral of the form $\int_\gamma \psi_i(t)\, t^{-x-1}\, dt$ like in the example given in the Introduction, we shall take a sum of integrals $\int_{C_\nu} \psi_{i,\,\nu}(t)\, t^{-x-1}\, dt$ where the C_ν are parts of a contour, and correct in some way for the differences between the $\psi_{i,\,\nu}$ (cf. formula (4.6)).

By J_i we denote the set

$$J_i := \{j \in \mathbb{Z}: k_j = k_i, \alpha_j = \alpha_i, |\lambda_j| < |\lambda_i|\}.$$

Lemma 4.1. *Let $i \in \mathbb{Z}$. There exist integers $M'_i < 1 \leqslant N_i < N'_i$ and, for every $\nu \in \{M'_i, ..., N'_i\}$, an open interval $I_{i,\nu} = (\alpha_{i,\nu}, \beta_{i,\nu})$, and a solution $\psi_{i,\nu}$ of (D), such that the following conditions are satisfied.*

(1) *$\{I_{i,\nu}\}_{\nu=M'_i}^{N'_i}$ is a good covering of $(\alpha_{i,M'_i}, \beta_{i,N'_i})$ and $\{I_{i,\nu}\}_{\nu=1}^{N_i}$ is a covering of $\tilde{I}_i = (\alpha_i - (3\pi/2k_i), \alpha_i - (\pi/2k_i))$.*

(2) *$\psi_{i,\nu}(t) \sim \hat{\psi}_i(t)$ as $t \to 0$, $\arg t \in I_{i,\nu}$, $\nu \in \{1, ..., N_i\}$.*

(3) *For $\nu = 0$ and $\nu = N_i + 1$ there exist solutions $\breve{\psi}_{j,\nu}$ with the property that $\breve{\psi}_{j,\nu} \sim \hat{\psi}_j$ as $t \to 0$, $\arg t \in I_{i,\nu}$, $j \in J_i \cup \{i\}$, complex numbers c_j, $j \in J_i$, and a solution $\tilde{\psi}_{i,\nu} \in \mathscr{A}^{<k_i}(I_{i,\nu})$ such that $\psi_{i,\nu}$ can be written in the following form*

$$\psi_{i,\nu} = \breve{\psi}_{i,\nu} + \sum_{j \in J_i} c_j \breve{\psi}_{j,\nu} + \tilde{\psi}_{i,\nu}. \tag{4.1}$$

(4) *$\psi_{i,\nu} \in \mathscr{A}^{<k_i}(I_{i,\nu})$ for all $\nu < 0$ and all $\nu > N_i + 1$.*

(5) *ψ_{i,M'_i} and ψ_{i,N'_i} have moderate growth as $t \to 0$, $\arg t \in (\alpha_{i,M'_i}, \alpha_{i,M'_i+1})$ and as $t \to 0$, $\arg t \in (\beta_{i,N'_i-1}, \beta_{i,N'_i})$, respectively. $\psi_{i,\nu} - \psi_{i,\nu+1}$ has moderate growth as $t \to 0$, $\arg t \in I_{i,\nu} \cap I_{i,\nu+1}$, $\nu \in \{M'_i, ..., N'_i - 1\}$.*

Proof. According to Lemma 2.4 there exist an integer N_i, open intervals $I_{i,\nu} = (\alpha_{i,\nu}, \beta_{i,\nu})$, $\nu = 0, ..., N_i + 1$, solutions $\psi_{i,\nu}$ of (D), $\nu \in \{1, ..., N_i\}$, and solutions $\breve{\psi}_{i,0}$ and $\breve{\psi}_{i,N_i+1}$, such that the following conditions are satisfied

- $\{I_{i,\nu}\}_{\nu=0}^{N_i+1}$ is a good covering of $[\alpha_i - (3\pi/2k_i), \alpha_i - (\pi/2k_i)]$.
- $\{I_{i,\nu}\}_{\nu=1}^{N_i}$ is a good covering of $(\alpha_i - (3\pi/2k_i), \alpha_i - (\pi/2k_i))$.
- $\psi_{i,\nu}(t) \sim \hat{\psi}_i(t)$ as $t \to 0$, $\arg t \in I_{i,\nu}$, $\nu \in \{1, ..., N_i\}$ and $\breve{\psi}_{i,\nu}(t) \sim \hat{\psi}_i(t)$ as $t \to 0$, $\arg t \in I_{i,\nu}$, for $\nu = 0$ and $\nu = N_i + 1$.
- $\psi_{i,\nu} - \psi_{i,\nu+1} \in \mathscr{A}_D^{<-k_i}(I_{i,\nu} \cap I_{i,\nu+1})$ for all $\nu \in \{1, ..., N_i - 1\}$.
- $e^{-p_i(t)}(\breve{\psi}_{i,0} - \psi_{i,1}) \in \mathscr{A}^{\leqslant -k_i}(I_{i,0} \cap I_{i,1})$ and

$$e^{-p_i(t)}(\breve{\psi}_{i,N_i+1} - \psi_{i,N_i}) \in \mathscr{A}^{\leqslant -k_i}(I_{i,N_i} \cap I_{i,N_i+1}). \tag{4.2}$$

Let J_i^- and J_i^+ denote the following sets of integers

$$J_i^- := \left\{ j \in \mathbb{Z}: 0 < k_j < k_i, \alpha_j - \frac{3\pi}{2k_j} \leqslant \alpha_i - \frac{3\pi}{2k_i} < \alpha_j - \frac{\pi}{2k_j} \right\}$$

$$J_i^+ := \left\{ j \in \mathbb{Z}: 0 < k_j < k_i, \alpha_j - \frac{3\pi}{2k_j} < \alpha_i - \frac{\pi}{2k_i} \leqslant \alpha_j - \frac{\pi}{2k_j} \right\}.$$

Note that $J_i^- \cup J_i$ consists of all integers j such that $e^{p_j(t)} \to \infty$ but $e^{p_j(t)-p_i(t)} \to 0$ as $t \to 0$, $\arg t \in (\alpha_i - (3\pi/2k_i), \alpha_i - (3\pi/2k_i) + \varepsilon)$ for some $\varepsilon > 0$. Similarly $J_i^+ \cup J_i$ consists of all integers j such that $e^{p_j(t)} \to \infty$ but $e^{p_j(t)-p_i(t)} \to 0$ as $t \to 0$, $\arg t \in (\alpha_i - (\pi/2k_i) - \varepsilon, \alpha_i - (\pi/2k_i))$ for some $\varepsilon > 0$. If $\beta_{i,0} - \alpha_{i,0}$ is sufficiently small, then, by Lemma 2.4, there exist integers $M_j^* < 0$, $j \in J_i^- \cup J_i$, open intervals $I_{i,v} := (\alpha_{i,v}, \beta_{i,v})$, $v \in \{\min_{j \in J_i^- \cup J_i} M_j^*, ..., -1\}$ such that $\alpha_i - (3\pi/2k_i) - (\pi/k_j) < \alpha_{i, M_j^*} < \beta_{i, M_j^*} \leqslant \alpha_j - (3\pi/2k_j)$ and $\{I_{i,v}\}_{v=M_j^*}^{0}$ is a good covering of $(\alpha_{i, M_j^*}, \alpha_i - (3\pi/2k_i)]$ and solutions $\psi_{j,v}$ of (D), $v = M_j^*, ..., 0$, with the following properties

$$\psi_{j,v}(t) \sim \hat{\psi}_j(t) \qquad \text{as} \quad t \to 0, \arg t \in I_{i,v}, v \in \{M_j^*, ..., 0\}$$

and

$$\psi_{j,v} - \psi_{j,v+1} \in \mathscr{A}_D^{\leqslant -k_j}(I_{i,v} \cap I_{i,v+1}), v \in \{M_j^*, ..., -1\}$$

Similarly, if $\beta_{i,N_i+1} - \alpha_{i,N_i+1}$ is sufficiently small, there exist integers $N_j^* > N_i + 1$, $j \in J_i^+ \cup J_i$, open intervals $I_{i,v} := (\alpha_{i,v}, \beta_{i,v})$, $v \in \{N_i + 2, ..., \max_{j \in J_i^+ \cup J_i} N_j^*\}$, such that $\alpha_j - (\pi/2k_j) \leqslant \alpha_{i, N_j^*} < \beta_{i, N_j^*} < \alpha_i - (\pi/2k_i) + (\pi/k_j)$, and $\{I_{i,v}\}_{v=N_i+1}^{N_j^*}$ is a good covering of $[\alpha_i - (\pi/2k_i), \beta_{i, N_j^*})$ and solutions $\psi_{j,v}$ of (D), $v = N_i + 1, ..., N_j^*$, with the property that

$$\psi_{j,v}(t) \sim \hat{\psi}_j(t) \qquad \text{as} \quad t \to 0, \quad \arg t \in I_{i,v}, v \in \{1, ..., N_j^*\}$$

and

$$\psi_{j,v} - \psi_{j,v+1} \in \mathscr{A}_D^{\leqslant -k_j}(I_{i,v} \cap I_{i,v+1}), v \in \{N_i + 1, ..., N_j^* - 1\}$$

From (4.2) we deduce that

$$\begin{aligned} \psi_{i,1} - \breve{\psi}_{i,0} &= \sum_{j \in J_i^- \cup J_i} c_j^- \psi_{j,0} + \psi^- \\ \psi_{i,N_i} - \breve{\psi}_{i,N_i+1} &= \sum_{j \in J_i^+ \cup J_i} c_j^+ \psi_{j,N_i+1} + \psi^+ \end{aligned} \tag{4.3}$$

where $c_j^-, c_j^+ \in \mathbb{C}$ and ψ^- and ψ^+ are solutions of (D) with moderate growth as $t \to 0$, $\arg t \in I_{i,0} \cap \tilde{I}_i$ and $\arg t \in \tilde{I}_i \cap I_{i,N_i+1}$, respectively (cf. Remark 2.2). We now define $\psi_{i,0}$ and ψ_{i,N_i+1} by

$$\begin{aligned} \psi_{i,0} &= \breve{\psi}_{i,0} + \sum_{j \in J_i^- \cup J_i} c_j^- \psi_{j,0} \\ \psi_{i,N_i+1} &= \breve{\psi}_{i,N_i+1} + \sum_{j \in J_i^+ \cup J_i} c_j^+ \psi_{j,N_i+1} \end{aligned} \tag{4.4}$$

It is obvious from (4.3) that $\psi_{i,0}-\psi_{i,1}$ has moderate growth as $t\to 0$, $\arg t\in I_{i,0}\cap I_{i,1}$ and $\psi_{i,N_i}-\psi_{i,N_i+1}$ has moderate growth as $t\to 0$, $\arg t\in I_{i,N_i}\cap I_{i,N_i+1}$. Let $M'_i:=\min_{j\in J_i^-\cup J_i} M_j^*$, $N'_i:=\max_{j\in J_i^+\cup J_i} N_j^*$. For $v\in\{M'_i, ..., -1\}$ we define

$$\psi_{i,v}=\sum_{j\in J_i^-:\, M_j^*\leqslant v} c_j^-\psi_{j,v}$$

and for $v\in\{N_i+2, ..., N'_i\}$,

$$\psi_{i,v}=\sum_{j\in J_i^+:\, N_j^*\geqslant v} c_j^+\psi_{j,v}$$

Then we have, for all $v>N_i+1$,

$$\psi_{i,v}-\psi_{i,v+1}=\sum_{j\in J_i^+:\, N_j^*\geqslant v+1} c_j^+(\psi_{j,v}-\psi_{j,v+1})+\sum_{j\in J_i^+:\, N_j^*=v} c_j^+\psi_{j,N_j^*}$$

As $I_{i,N_j^*}\subset I_j$ for all $j\in J_i^+$, we conclude that $\psi_{j,N_j^*}\sim 0$ as $t\to 0$, $\arg t\in I_{i,N_j^*}$. It follows easily that $\psi_{i,v}-\psi_{i,v+1}$ has moderate growth as $t\to 0$, $\arg t\in I_{i,v}\cap I_{i,v+1}$ for all $v>N_i+1$. In a similar manner one proves that $\psi_{i,v}-\psi_{i,v+1}$ has moderate growth as $t\to 0$, $\arg t\in I_{i,v}\cap I_{i,v+1}$ for $v=N_i+1$ and for all $v<0$. The remaining statements of the lemma are also easily verified. ∎

Remark 4.2. The collection of functions $\{\psi_{i,v}\}_{v=M'_i}^{N'_i}$ defines an element $\Psi_i\in H_c^0(\mathbb{R}, \mathscr{A}_D^{\leqslant k_i}/\mathscr{A}_D^{\leqslant 0})$ (i.e. a section of $\mathscr{A}_D^{\leqslant k_i}/\mathscr{A}_D^{\leqslant 0}$ with compact support) with the following properties:

$$\operatorname{supp}\Psi_i\subset\left[\min\left\{\alpha_j-\frac{3\pi}{2k_j}: j\in J_i^-\cup J_i\cup i: c_j^-\neq 0\right\},\right.$$
$$\left.\max\left\{\alpha_j-\frac{\pi}{2k_j}: j\in J_i^+\cup J_i\cup i: c_j^+\neq 0\right\}\right]$$

where $c_i^-=c_i^+=1$, and

$$\operatorname{supp}\Psi_i(\operatorname{mod}\mathscr{A}^{<k_i})\subset\left[\alpha_i-\frac{3\pi}{2k_i}, \alpha_i-\frac{\pi}{2k_i}\right]$$

With the aid of Remark 2.5 we can prove a stronger statement, viz. the existence of sections Ψ_i^+ and $\Psi_i^-\in H_c^0(\mathbb{R}, \mathscr{A}_D^{\leqslant k_i}/\mathscr{A}_D^{\leqslant 0})$ such that

$$\operatorname{supp}\Psi_i^+\subset\left[\alpha_i-\frac{3\pi}{2k_i}, \max\left\{\alpha_j-\frac{\pi}{2k_j}: j\in J_i^+\cup J_i\cup i: c_j^+\neq 0\right\}\right]$$

and

$$\operatorname{supp} \Psi_i^- \subset \left[\min \left\{ \alpha_j - \frac{3\pi}{2k_j} : j \in J_i^- \cup J_i \cup i : c_j^- \neq 0 \right\}, \alpha_i - \frac{\pi}{2k_i} \right]$$

DEFINITION 4.3. Let $i \in \mathbb{Z}$ and suppose we are given open intervals $I_{i,\nu}$ and solutions $\psi_{i,\nu}$ of (D), $\nu = M_i', ..., N_i'$, such that the conditions of Lemma 4.1 are satisfied. Furthermore, let

$$I_{i,M_i'-1} = (-\infty, \alpha_{i,M_i'+1}), \qquad I_{i,N_i'+1} = (\beta_{i,N_i'-1}, \infty)$$

and

$$\psi_{i,M_i'-1} = \psi_{i,N_i'+1} = 0$$

Let γ_i^ν denote a half line from O to ∞ in $S(I_{i,\nu} \cap I_{i,\nu+1})$, $\nu = M_i' - 1, ..., N_i'$. We define

$$\tilde{y}_i = \sum_{\nu = M_i'-1}^{N_i'} \mathscr{P}_{\gamma^\nu_i}(\psi_{i,\nu} - \psi_{i,\nu+1}) \tag{4.5}$$

Remark 4.4. Using the notation of Definition 4.3, let $R > 0$ and, for each $\nu \in \{M_i' - 1, ..., N_i'\}$, let θ_ν denote the direction of γ_i^ν. For $\nu \in \{M_i', ..., N_i'\}$ let C_ν denote the arc $C_\nu := \{t : |t| = R, \theta_{\nu-1} < \arg t < \theta_\nu\}$, described in the positive sense. By means of deformation of paths of integration it is easily shown that (4.5) is equivalent to

$$\begin{aligned} \tilde{y}_i(x) = & \sum_{\nu = M_i'-1}^{N_i'} \int_0^{\mathrm{R}e^{i\theta_\nu}} (\psi_{i,\nu}(t) - \psi_{i,\nu+1}(t))\, t^{-x-1}\, dt \\ & - \sum_{\nu = M_i'}^{N_i'} \int_{C_\nu} \psi_{i,\nu}(t)\, t^{-x-1}\, dt \end{aligned} \tag{4.6}$$

provided $\operatorname{Re} x$ is sufficiently small. This definition is more general than the previous one, as it depends only on the local properties of the functions $\psi_{i,\nu}$ at the origin. It can be used even if (D) has an irregular singularity at ∞ or if it has other singular points besides O and ∞. In the latter case, the number R has to be chosen smaller than the least absolute value of these singular points.

It can be verified that, for each $i \in \mathbb{Z}$, $\tilde{y}_i$ is a solution of (Δ). From (4.6) one deduces that it is analytic in a left half plane and meromorphic in $\mathbb{C}$.

THEOREM 4.5. *For each* $i \in \mathbb{Z}$, $\tilde{y}_i(x) \sim \hat{y}_i(x)$, *as* $x \to \infty$ *in* $S(\pi/2, 3\pi/2)$. *Consequently, the set* $\tilde{Y} := \{\tilde{y}_i : i \in \{1, ..., M\}\}$ *is a fundamental system of solutions of* (Δ).

Proof. To simplify the argument, we assume that $q=1$, $\lambda_i=1$, $s_i=0$, $\rho_i=-\frac{1}{2}k_i$ and $J_i=\varnothing$ (cf. Remark 4.7). Furthermore, we drop the subscript i. Let

$$G:=\mathscr{P}_{\gamma^{N+1}}(\exp p(t))-\mathscr{P}_{\gamma^{-1}}(\exp p(t)) \tag{4.7}$$

$G(x)$ admits an asymptotic representation of the form

$$\hat{G}(x):=x^{x/k}\exp \tilde{p}(x)\,\hat{d}(x) \tag{4.8}$$

where $\hat{d}\in\mathbb{C}[\![x^{-1/k}]\!]$, as $x\to\infty$ in $S(\pi/2, 3\pi/2)$ (cf. [4]). It is easily seen that

$$\hat{G}(x)^{-1}\,\hat{G}(x-1)\in x^{-1/k}\mathbb{C}[\![x^{-1/k}]\!] \tag{4.9}$$

Let $\hat{g}(t)=\sum_{l=0}^{\infty}g_l t^l$, where $\hat{g}$ $(=\hat{g}_i)$ is defined in Definition 2.1. From (4.8) and (4.9) we deduce the existence of complex numbers h_l, $l\in\mathbb{N}\cup\{0\}$, such that, for all $n\in\mathbb{N}$ the following estimate holds

$$\sum_{l=0}^{n-1}\{g_l G(x-l)\,x^{-x/k}\exp(-\tilde{p}(x))-h_l x^{-l/k}\}=O(x^{-n/k})$$

as $x\to\infty$ in $S(\pi/2, 3\pi/2)$. Moreover, the numbers h_l coincide with the coefficients of $\hat{h}_i$ in (3.2). Thus it suffices to show that, for all $n\in\mathbb{N}$,

$$\left\{\tilde{y}(x)-\sum_{l=0}^{n-1}g_l G(x-l)\right\}x^{-x/k}\exp(-\tilde{p}(x))=O(x^{-n/k}) \tag{4.10}$$

as $x\to\infty$ in $S(\pi/2, 3\pi/2)$. From now on we consider a fixed value of n. Let ψ_ν, $\nu=M'-1, ..., N'+1$, denote the functions used in Definition 4.3. For all $\nu\in\{M'-1, ..., N'+1\}$ we define

$$\psi_\nu^*(t):=\begin{cases}\psi_\nu(t) & \text{if}\quad \nu<0\quad\text{or}\quad \nu>N+1\\ \psi_\nu(t)-\exp(p(t))\sum_{l=0}^{n-1}g_l t^l & \text{if}\quad 0\leqslant\nu\leqslant N+1\end{cases}$$

Note that $\psi_\nu^*-\psi_{\nu+1}^*=\psi_\nu-\psi_{\nu+1}$ for all $\nu\notin\{-1, N+1\}$, whereas

$$\psi_{-1}^*-\psi_0^*=\psi_{-1}-\psi_0+e^{p(t)}\sum_{l=0}^{n-1}g_l t^l$$

and

$$\psi_{N+1}^*-\psi_{N+2}^*=\psi_{N+1}-\psi_{N+2}-e^{p(t)}\sum_{l=0}^{n-1}g_l t^l$$

With (4.6), (4.8) and (4.9) it now follows that (4.10) is equivalent to

$$\sum_{v=M_i'-1}^{N_i'} \int_0^{Re^{i\theta_v}} (\psi_v^*(t) - \psi_{v+1}^*(t))\, t^{-x-1}\, dt - \sum_{v=M_i'}^{N_i'} \int_{C_v} \psi_v^*(t)\, t^{-x-1}\, dt$$
$$= x^{x/k} \exp \tilde{p}(x) \cdot O(x^{-n/k}) \tag{4.11}$$

for some $R>0$, as $x \to \infty$ in $S(\pi/2, 3\pi/2)$. The estimate (4.11) can be deduced from Lemma 4.6 below. ∎

LEMMA 4.6. *Let $i \in \mathbb{Z}$, $n \in \mathbb{N} \cup \{0\}$ and $R>0$. Suppose we are given intervals $I_{i,v} = (\alpha_{i,v}, \beta_{i,v})$ and functions ψ_v^* analytic in $S_{i,v} := S(I_{i,v}, R)$, $v = M'_i, ..., N'_i$, such that conditions* (1), (3) *and* (4) *of Lemma* 4.1, *with $\psi_{i,v}$ replaced by ψ_v^* are satisfied. Assume that, in addition, the following conditions hold.*

(i) *for all $v \in \{1, ..., N_i\}$, $\psi_v^*(t) = O(t^n e^{p_i(t)})$ as $t \to 0$ in $S_{i,v}$.*

(ii) *Let $S_i^- := S_{i,0}$ and $S_i^+ := S_{i,N_i+1}$. There exist analytic functions ψ^-, $\tilde{\psi}^-$, ψ^+ and $\tilde{\psi}^+$ with the following properties:*

1. $\psi^- + \tilde{\psi}^- = \psi_0^*$,
2. $\psi^+ + \tilde{\psi}^+ = \psi_{N_i+1}^*$,
3. $\psi^\pm(t) = O(t^n e^{p_i(t)})$ *as $t \to 0$ in $S_i^\pm$*,
4. $|\tilde{\psi}^\pm(t)| \leqslant C \exp(|t|^{-k_i+\delta})$ *for all $t \in S_i^\pm$, where $C>0$ and $\delta>0$.*

Furthermore, let $\psi_{M_i'-1}^* = \psi_{N_i'+1}^* = 0$. Then the function y^* defined by the left-hand side of (4.11) *satisfies the following estimate*

$$y^*(x) = O(x^{(x-n)/k} e^{\tilde{p}_i(x)}) \qquad as \quad x \to \infty \ in\ S\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \tag{4.12}$$

uniformly on closed subsectors.

Proof. We again assume that $q=1$ and $\lambda_i = 1$ and drop the subscript i. Let

$$F(x) := x^{x/k} \exp \tilde{p}(x)$$

in view of the fact that $F(x-n) = F(x)\, O(x^{-n/k})$ as $x \to \infty$ it suffices to prove the lemma for $n=0$.

By $t(x)$ we denote the unique saddle point of the function $t^{-x} e^{p(t)}$ belonging to $S(-(3\pi/2k), -(\pi/2k))$ for sufficiently large $|x|$ and we put

$$|t(x)| = r_x, \qquad \arg t(x) = \theta(x)$$

From the fact that

$$t(x) = \left(\frac{x}{k}\right)^{-1/k} (1 + o(1)) \qquad \text{as} \quad x \to \infty \tag{4.13}$$

and the definition of $\tilde{p}$ implied by (3.1) and (3.2), we deduce that

$$F(x) = t(x)^{-x}\, e^{p(t(x)) + O(1)} = \left(\frac{x}{k}\right)^{x/k} e^{(-x/k + o(x))} \qquad \text{as} \quad x \to \infty \tag{4.14}$$

We begin by proving that (4.12) holds as $x \to \infty$ in a fixed direction, close to $3\pi/2$. Let θ_ν denote the direction of γ^ν, $\nu \in \{M', ..., N'\}$. We choose a small positive number $\varepsilon < \frac{2}{3}k\theta_0 + \pi$ and take $\arg x = (3\pi/2) - \varepsilon$. Due to (4.13) we have

$$-\frac{3\pi}{2} + \frac{\varepsilon}{2} < k\theta(x) < k\theta_0 - \frac{\varepsilon}{2} \tag{4.15}$$

provided $|x|$ is sufficiently large. Let $d \in (1, (k/k - \delta))$. For all $\nu \in \{M' - 1, ..., N'\}$, let

$$a_\nu := r_x e^{i\theta_\nu}, \qquad b_\nu := r_x^d e^{i\theta_\nu}$$

For all $\nu \geqslant 0$ γ_*^ν will denote the segment from O to a_ν and for all $\nu < 0$ the segment from O to b_ν. Furthermore, we define

$$C_\nu^* = \begin{cases} \{t: \theta_{\nu-1} < \arg t < \theta_\nu,\ |t| = r_x\} & \text{if} \quad 1 \leqslant \nu \leqslant N' \\ \{t: \theta_{\nu-1} < \arg t < \theta_\nu,\ |t| = r_x^d\} & \text{if} \quad M' \leqslant \nu < 1 \end{cases}$$

and

$$\phi_\nu := \psi_\nu^* - \psi_{\nu+1}^* \qquad \text{if} \quad \nu \in \{M', ..., N' - 1\},$$

$$\phi_{M'-1} := -\psi_1^* \qquad \text{and} \qquad \phi_{N'} := \psi_{N'}^*$$

Note that $C_\nu^* \subset S_\nu$ for all $\nu \in \{M', ..., N'\}$. For all $x \in S(\pi/2, 3\pi/2)$ the function y^* may be represented in the following manner

$$y^*(x) = \sum_{\nu = M'-1}^{N'} \int_{\gamma_*^\nu} \phi_\nu(t)\, t^{-x-1}\, dt - \sum_{\nu = M'}^{N'} \int_{C_\nu^*} \psi_\nu^*(t)\, t^{-x-1}\, dt$$
$$+ \int_{a_0}^{b_0} \psi_0^*(t)\, t^{-x-1}\, dt$$

where the arcs $C_{M'}^*, ..., C_{N'}^*$ are described in the positive sense.

We begin by estimating the integrals

$$I^{\nu}(x) := \int_{\gamma_*^{\nu}} \phi_{\nu}(t)\, t^{-x-1}\, dt, \qquad \nu = M'-1, ..., N'.$$

By assumption, ϕ_{ν} has moderate growth as $t \to 0$ on γ^{ν} for all $\nu \in \{M'-1, ..., N'\}$. Consequently, there exist positive numbers K and τ such that, for all $\nu \in \{M'-1, ..., N'\}$,

$$|\phi_{\nu}(t)| \leqslant K\, |t|^{\tau}, \qquad t \in \gamma^{\nu}, \quad |t| < 1.$$

Hence we deduce that, for $\nu \geqslant 0$ and $\operatorname{Re} x < -\tau$,

$$|I^{\nu}(x)| \leqslant K \exp(\theta_{\nu} \operatorname{Im} x)(-\operatorname{Re} x + \tau)^{-1}\, r_x^{-\operatorname{Re} x + \tau}$$

With (4.13) it follows that, for $\nu \geqslant 0$,

$$I^{\nu}(x) \leqslant \exp\left\{\operatorname{Re}\left(\frac{x}{k} \log \frac{x}{k}\right) + \left(\frac{1}{k} \arg x + \theta_{\nu}\right) \operatorname{Im} x + o(x)\right\}$$

as $x \to \infty$ in $S(\pi/2, 3\pi/2)$. Noting that $\operatorname{Im} x \to -\infty$ as $|x| \to \infty$, $\arg x = (3\pi/2) - \varepsilon$, and $\theta_{\nu} \geqslant \theta_0 > -(1/k) \arg x$ and using (4.14), we conclude that, for all $\nu \geqslant 0$,

$$I^{\nu}(x) = o(F(x)) \qquad \text{as} \quad x \to \infty, \quad \arg x = \frac{3\pi}{2} - \varepsilon \tag{4.16}$$

For $\nu < 0$ we have

$$|I^{\nu}(x)| \leqslant K \exp(\theta_{\nu} \operatorname{Im} x)(-\operatorname{Re} x + \tau)^{-1}\, r_x^{d(-\operatorname{Re} x + \tau)}$$

Hence, due to (4.13),

$$I^{\nu}(x) \leqslant \exp\left\{\frac{d}{k} \operatorname{Re} x \log |x| + O(x)\right\} \qquad \text{as} \quad x \to \infty, \quad \arg x = \frac{3\pi}{2} - \varepsilon$$

As $d > 1$ and $\cos(3\pi/2 - \varepsilon) < 0$, it follows that (4.16) holds for all $\nu \in \{M'-1, ..., N'\}$.

Now consider the function $g(\theta, x)$ defined by

$$g(\theta, x) = \operatorname{Re}\{-x \log(r_x e^{i\theta}) + p(r_x e^{i\theta})\}.$$

With the aid of (4.13) we find that

$$\frac{\partial g}{\partial \theta}(\theta, x) = |x|\, (\sin k\theta + \sin \arg x) + o(x) = |x|\, (\sin k\theta - \cos \varepsilon) + o(x) \tag{4.17}$$

and

$$\frac{\partial^2 g}{\partial \theta^2}(\theta, x) = k\,|x| \cos k\theta + o(x) \tag{4.18}$$

as $x \to \infty$, $\arg x = 3\pi/2 - \varepsilon$. Moreover, the definition of $t(x)$ implies that

$$\frac{\partial g}{\partial \theta}(\theta(x), x) = 0 \tag{4.19}$$

From (4.17)–(4.19) and (4.15) we infer that $\partial g/\partial\theta > 0$ on $[-(3\pi + \varepsilon/2k), -(3\pi - \varepsilon/2k)]$, $(\partial g/\partial\theta) < 0$ on $[-\pi/k, 0]$ whereas $(\partial^2 g/\partial\theta^2) < 0$ on $[-(3\pi - \varepsilon/2k), -(\pi + \varepsilon/2k)]$ for sufficiently large values of $|x|$. Consequently

$$g(\theta, x) \leqslant g(\theta(x), x) \qquad \text{for all} \quad \theta \in \left[-\frac{3\pi + \varepsilon}{2k}, -\frac{\pi}{2k}\right]$$

provided $|x|$ is sufficiently large. With (4.14) it follows that

$$\int_{-(3\pi + \varepsilon/2k)}^{-(\pi/2k)} e^{g(\theta, x)}\, d\theta = O(F(x)) \qquad \text{as} \quad x \to \infty, \quad \arg x = -\frac{3\pi}{2} + \varepsilon. \tag{4.20}$$

For all $\nu \in \{1, ..., N'\}$ we have

$$\left|\int_{C_\nu^*} \exp(|t|^{-k+\delta})\, t^{-x-1}\, dt\right| \leqslant |\mathrm{Im}\, x|^{-1} \exp\{r_x^{-k+\delta} + \theta_{\nu-1}\,\mathrm{Im}\, x\}\, r_x^{-\mathrm{Re}\, x}.$$

Hence, due to (4.13),

$$\int_{C_\nu^*} \exp\,(|t|^{-k+\delta})\, t^{-x-1}\, dt$$
$$\leqslant \exp\left\{\mathrm{Re}\left(\frac{x}{k}\log\frac{x}{k}\right) + \left(\frac{1}{k}\arg x + \theta_{\nu-1}\right)\mathrm{Im}\, x + o(x)\right\}. \tag{4.21}$$

From (4.20), (4.21) and (4.14) we deduce that

$$\sum_{\nu=1}^{N'} \int_{C_\nu^*} \psi_\nu^*(t)\, t^{-x-1}\, dt = O(F(x)) \qquad \text{as} \quad x \to \infty, \arg x = \frac{3\pi}{2} - \varepsilon$$

For all $\nu \in \{M', ..., 0\}$ we have

$$\left|\int_{C_\nu^*} \exp(|t|^{-k+\delta})\, t^{-x-1}\, dt\right| \leqslant |\mathrm{Im}\, x|^{-1} \exp\{r_x^{-d(k-\delta)} + \theta_{\nu-1}\,\mathrm{Im}\, x\}\, r_x^{-d\,\mathrm{Re}\, x}$$

Using (4.13) and noting that $d(k-\delta)<k$, we find

$$\int_{C_\nu^*} \exp(|t|^{-k+\delta})\, t^{-x-1}\, dt \leqslant \exp\left\{\frac{d}{k} \operatorname{Re} x \log|x| + O(x)\right\}$$

as $x \to \infty$ in $S(\pi/2, 3\pi/2)$. Hence it follows that

$$\sum_{\nu=M'}^{-1} \int_{C_\nu^*} \psi_\nu^*(t)\, t^{-x-1}\, dt + \int_{C_0^*} \tilde{\psi}^-(t)\, t^{-x-1}\, dt = o(F(x))$$

as $x \to \infty$ in $S(\pi/2, 3\pi/2)$. Furthermore, we have

$$\left|\int_{a_0}^{b_0} \exp(|t|^{-k+\delta})\, t^{-x-1}\, dt\right| \leqslant \exp(r_x^{-d(k-\delta)} + \theta_0 \operatorname{Im} x)\, |\operatorname{Re} x|^{-1}\, r_x^{-\operatorname{Re} x}$$

Due to (4.13) and the fact that $d(k-\delta)<k$, this implies that

$$\int_{a_0}^{b_0} \tilde{\psi}^-(t)\, t^{-x-1}\, dt \leqslant \exp\left\{\operatorname{Re}\left(\frac{x}{k} \log\frac{x}{k}\right) + \left(\frac{1}{k} \arg x + \theta_0\right) \operatorname{Im} x + o(x)\right\}$$

$$= o(F(x)) \qquad \text{as} \quad x \to \infty, \quad \arg x = \frac{3\pi}{2} - \varepsilon.$$

It remains to be shown that

$$\int_{C_0^*} \psi^-(t)\, t^{-x-1}\, dt - \int_{a_0}^{b_0} \psi^-(t)\, t^{-x-1}\, dt$$

$$= O(F(x)) \qquad \text{as} \quad x \to \infty, \quad \arg x = \frac{3\pi}{2} - \varepsilon. \tag{4.22}$$

Let $a^- = r_x e^{(-i(3\pi + \varepsilon/2k))}$. As ψ^- is analytic in S^- the left-hand side of (4.22) is equal to

$$\int_{a_-}^{a_0} \psi^-(t)\, t^{-x-1}\, dt - \int_{a_-}^{b_{-1}} \psi^-(t)\, t^{-x-1}\, dt. \tag{4.23}$$

From (4.20) we infer that the first term in (4.23) is $O(F(x))$ as $x \to \infty$, $\arg x = (3\pi/2) - \varepsilon$ and it is easily verified that the same is true of the second term. Thus we find that

$$y^*(x) = O(F(x)) \qquad \text{as} \quad x \to \infty, \quad \arg x = \frac{3\pi}{2} - \varepsilon.$$

In an analogous manner it can be shown that

$$y^*(x) = O(F(x)) \qquad \text{as} \quad x \to \infty, \quad \arg x = \frac{\pi}{2} + \varepsilon.$$

It is easily seen that y^* has at most exponential growth as $x \to \infty$ in $S(\pi/2, 3\pi/2)$. Hence it follows that there exist positive numbers R and C such that

$$|F(x)^{-1}\, y^*(x)| \leqslant C \exp(C|x| \log |x|)$$

if $x \in S((\pi/2) + \varepsilon, (3\pi/2) - \varepsilon)$ and $|x| \geqslant R$. Applying a Phragmén–Lindelöf-type of argument (cf. [11]) we conclude that $y^*(x) = O(F(x))$ as $x \to \infty$, uniformly on $\arg x \in [(\pi/2) + \varepsilon, (3\pi/2) - \varepsilon]$. As ε is an arbitrarily small positive number it follows that $y^*(x) = O(F(x))$ as $x \to \infty$ in $S((\pi/2), (3\pi/2))$, uniformly on closed subsectors. ▮

Remark 4.7. In order to see that, under the assumption that $J_i = \varnothing$, made at the beginning of the proof of Theorem 4.5, condition (ii) of Lemma 4.6 is satisfied, write $\psi_{i,0}$ and ψ_{i, N_i+1} in the form (4.1) and put

$$\check{\psi}_{i,0}(t) - e^{p_i(t)} \sum_{l=0}^{n-1} g_{il} t^l = \psi^-(t)$$

where g_{il} denotes the coefficient of t^l in $\hat{g}_i$,

$$\check{\psi}_{i, N_i+1}(t) - e^{p_i(t)} \sum_{l=0}^{n-1} g_{il} t^l = \psi^+(t),$$

$$\tilde{\psi}_{i,0}(t) = \tilde{\psi}^-(t),$$

and

$$\tilde{\psi}_{i, N_i+1}(t) = \tilde{\psi}^+(t).$$

If $J_i \neq \varnothing$ for some $i \in \mathbb{Z}$, we need a slight generalization of Lemma 4.6, which takes into account the contributions from $\psi_{j,0}$ and ψ_{j, N_i+1}, $j \in J_i$ in (4.4). The presence of such terms in (4.4) requires a particular choice of paths of integration in the proof of this lemma in order to obtain the desired estimates.

5. The Relation between the Fundamental Systems Y and $\tilde{Y}$

In the remaining part of this section we study the relation between two fundamental systems Y and $\tilde{Y}$ of the types defined in (3.7) and

Theorem 4.5, respectively. (Note that there is a considerable degree of freedom in the choice of both systems.) More generally, we consider the following problem. Let $\theta \in \mathbb{R}$ and let γ denote the half line from O to ∞ with direction θ. Let ψ be a solution of (D) with moderate growth as $t \to 0$ in a sector containing γ. Then the function $\mathscr{P}_\gamma(\psi)$ is a solution of $(\varDelta)$, meromorphic in $\mathbb{C}$. Hence there exist periodic functions π_i, $i = 1, ..., M$, of period 1, such that

$$\mathscr{P}_\gamma(\psi) = \sum_{i=1}^{M} \pi_i y_i. \tag{5.1}$$

We are interested in the dependence of the periodic functions π_i upon the solution ψ of (D). There are two possibilities: either $\psi \in (\mathscr{A}_D^{<0})_\theta$ (i.e., there is an open interval $I \ni \theta$ such that $\psi \in \mathscr{A}_D^{<0}(I)$), or $\psi(t)$ admits an asymptotic representation $\hat{\psi}$ of the form

$$\hat{\psi}(t) = \sum_{j:\, k^j = 0} c_j \hat{\psi}^j(t) \tag{5.2}$$

as $t \to 0$ in a sector containing γ, where $c_j \in \mathbb{C}$ and at least one of these complex numbers does not vanish. We begin by considering the first situation. Note that, in this case, $\mathscr{P}_\gamma(\psi)$ is analytic in a right half plane (cf. Proposition 1.1).

For all $i \in \mathbb{Z}$, let $I_{i,\nu}$ and $\psi_{i,\nu}$, $\nu = 1, ..., N_i$, denote the intervals and functions, respectively, mentioned in Definition 3.1. For every $i \in \mathbb{Z}$ we have fixed an integer $\nu_i \in \{1, ..., N_i\}$ and put $\psi_{i,\nu_i} = \psi_i$ and $y_{i,\nu_i} = y_i$. Let $\gamma_i := \gamma_{i,\nu_i}$. Thus we have

$$y_i = \mathscr{P}_{\gamma_i}(\psi_i).$$

PROPOSITION 5.1. *Let $\theta \in \mathbb{R}$ and let $k \in \{k^1, ..., k^m\}$, $k > 0$. There exists a finite set $J_k(\theta)$ of integers i with the property that $k_i \geqslant k$, such that every $\phi \in (\mathscr{A}_D^{\leqslant -k})_\theta$ can be written in the following form*

$$\phi = \sum_{i \in J_k(\theta)} c_i \psi_{i\theta}, \qquad c_i \in \mathbb{C}. \tag{5.3}$$

Moreover, if γ is the half line from O to ∞ with direction θ, we have

$$\mathscr{P}_\gamma(\phi) = \sum_{i \in J_k(\theta)} c_i y_i. \tag{5.4}$$

Proof. For every $j \in \{1, ..., m\}$ with the property that $\operatorname{Re} p^j(t) \to -\infty$ as $t \to 0$, $\arg t = \theta$, there is exactly one integer i such that $p_i = p^j$ and $\theta \in I_i$.

For every $i \in \mathbb{Z}$ such that $\theta \in I_i$ let μ_i denote an integer $\in \{1, ..., N_i\}$ with the property that $\theta \in I_{i, \mu_i}$. The germs at θ of the solutions ψ_{i, μ_i}, $i \in \mathbb{Z}$: $I_i \ni \theta$, $k_i \geqslant k$, form a basis of $(\mathscr{A}_D^{\leqslant -k})_\theta$ (cf. Remark 2.2). We shall prove the proposition by means of induction on $k^m - k$. Suppose that $k = k^m$. Let $J = \{i \in \mathbb{Z}: k_i = k^m$ and $\theta \in I_i\}$. Due to the fact that $(\mathscr{A}_D^{< -k^m})_\theta = \{0\}$, the conditions (ii) and (iii) of Lemma 2.4 imply that $\psi_i(t) \sim \hat{\psi}_i(t)$ as $t \to 0$ in S_i, for all $i \in J$. Thus the solutions $\psi_{i\theta}$, $i \in J$, form a basis of $(\mathscr{A}_D^{\leqslant -k})_\theta$. This proves the first statement of the proposition. The second statement follows by observing that

$$\mathscr{P}_\gamma(\psi_i) = \mathscr{P}_{\gamma_i}(\psi_i) = y_i \qquad \text{for all} \quad i \in J.$$

Now suppose that $k < k^m$ and that the statements of Proposition 5.1 are true for all $k' > k$. Let $J = \{i \in \mathbb{Z}: k_i = k$ and $\theta \in I_i\}$. Obviously, there exist complex numbers c_i, $i \in J$, and a function $\psi \in (\mathscr{A}_D^{< -k})_\theta$ such that

$$\phi = \sum_{j \in J} c_i \psi_{i, \mu_i} + \psi. \tag{5.5}$$

Furthermore, we have

$$\psi_{i, \mu_i} - \psi_i = \begin{cases} \sum_{v = \mu_i}^{v_i - 1} (\psi_{i, v} - \psi_{i, v+1}) & \text{if} \quad \mu_i < v_i \\ -\sum_{v = v_i}^{\mu_i - 1} (\psi_{i, v} - \psi_{u, v+1}) & \text{if} \quad \mu_i > v_i. \end{cases} \tag{5.6}$$

According to Lemma 2.4, condition (iii), $\psi_{i, v} - \psi_{i, v+1} \in \mathscr{A}_D^{< -k}(I_{i, v} \cap I_{i, v+1})$ for all $v \in \{1, ..., N_i - 1\}$, $i \in J$. Therefore, the first statement of the proposition follows from (5.5) and (5.6) through the use of the induction hypothesis.

From (5.5) we deduce that

$$\mathscr{P}_\gamma(\phi) = \sum_{i \in J} c_i \mathscr{P}_{\gamma_{i, \mu_i}}(\psi_{i, \mu_i}) + \mathscr{P}_\gamma(\psi) \tag{5.7}$$

for all $v \in \{1, ..., N_i - 1\}$ we have

$$\mathscr{P}_{\gamma_{i, v}}(\psi_{i, v}) - \mathscr{P}_{\gamma_{i, v+1}}(\psi_{i, v+1}) = \mathscr{P}_{\gamma_i^v}(\psi_{i, v} - \psi_{i, v+1})$$

where γ_i^v denotes a half line from O to ∞ in $S(I_{i, v} \cap I_{i, v+1})$. Hence

$$\mathscr{P}_{\gamma_{i, \mu_i}}(\psi_{i, \mu_i}) - \mathscr{P}_{\gamma_i}(\psi_i) = \begin{cases} \sum_{v = \mu_i}^{v_i - 1} \mathscr{P}_{\gamma_i^v}(\psi_{i, v} - \psi_{i, v+1}) & \text{if} \quad \mu_i < v_i \\ -\sum_{v = v_i}^{\mu_i - 1} \mathscr{P}_{\gamma_i^v}(\psi_{i, v} - \psi_{i, v+1}) & \text{if} \quad \mu_i > v_i \end{cases} \tag{5.8}$$

The second statement of the proposition now follows from (5.7) and (5.8) with the aid of the induction hypothesis. ∎

If $\psi \notin (\mathscr{A}_D^{<0})_\theta$, the situation is slightly more complicated, due to the fact that $\mathscr{P}_\gamma(\psi)$ now has poles at the points $\rho^j + n/q$, where $j \in \{1, ..., m\}$ such that $k^j = 0$ and the constant c_j in (5.2) is non-zero, and n is an integer with the property that the coefficient g^j_{nh} of $t^{n/q}(\log t)^h$ in $\hat{g}^j$ does not vanish for some value of h (cf. Proposition 1.1). For each $j \in \{1, ..., m\}$ there exists a solution $\psi^{j\theta}$ of (D) with the property that $\psi^{j\theta}(t) \sim \hat{\psi}^j(t)$ as $t \to 0$, $\arg t \in I_\theta := (\theta - (\pi/2k^m), \theta + (\pi/2k^m))$ (cf. Remark 2.2). Let $\mathscr{J}_0$ denote the set of $j \in \{1, ..., m\}$ such that $k^j = 0$. For all $j \in \mathscr{J}_0$ we can define a meromorphic solution $y^{j\theta}$ of (Δ) by

$$y^{j\theta} = \mathscr{P}_{\gamma_\theta}(\psi^{j\theta})$$

where γ_θ is the half line from O to ∞ with direction θ. (In general, this solution has an infinite number of poles in both a right and a left half plane.) Let $\theta_1, \theta_2 \in \mathbb{R}$ such that $I_{\theta_1} \cap I_{\theta_2} \neq \emptyset$. Obviously,

$$\psi^{j\theta_1} - \psi^{j\theta_2} \in \mathscr{A}_D^{<0}(I_{\theta_1} \cap I_{\theta_2}).$$

In view of Proposition 5.1, there is a finite set $J(\theta_1, \theta_2)$ of integers such that

$$\psi^{j\theta_1} - \psi^{j\theta_2} = \sum_{i \in J(\theta_1, \theta_2)} c_{ji} \psi_i,$$

where $c_{ji} \in \mathbb{C}$, and

$$y^{j\theta_1} - y^{j\theta_2} = \sum_{i \in J(\theta_1, \theta_2)} c_{ji} y_i.$$

In order to simplify the discussion, let us first assume that $q = 1$ and $s^j = 0$ for all $j \in \{1, ..., m\}$. Then we can choose the functions $\psi^{j\theta}$ in such a manner that

$$\psi^{j(\theta + 2\pi)}(t) = e^{2\pi i \rho^j} \psi^{j\theta}(te^{-2\pi I}).$$

Hence we deduce that

$$y^{j(\theta + 2\pi)}(x) = e^{-2\pi i(x - \rho^j)} y^{j\theta}(x). \tag{5.9}$$

With (5.9) we obtain the following result.

LEMMA 5.2. *Let $\theta \in \mathbb{R}$. Under the assumption made above, there exists a finite set of integers J_θ such that, for every $j \in \mathscr{J}_0$,*

$$\psi^{j\theta} - \psi^{j(\theta + 2\pi)} = \sum_{i \in J_\theta} c_{ji} \psi_i \tag{5.10}$$

where $c_{ji} \in \mathbb{C}$. Furthermore,

$$y^{j\theta} = \{1 - e^{-2\pi i(x - \rho^j)}\}^{-1} \sum_{i \in J_\theta} c_{ji} y_i(x). \tag{5.11}$$

Remark 5.3. The set J_θ need not be a subset of $\{1, ..., M\}$. However, under the assumption made above, we can choose the intervals I_{i,ν_i} and the functions ψ_i, $i \in \mathbb{Z}$, in such a way that

$$I_{i+M,\, \nu_{i+M}} = e^{2\pi i} I_{i,\nu_i} \qquad \text{and} \qquad \psi_{i+M}(t) = e^{2\pi i \rho_i} \psi_i(t e^{-2\pi I}).$$

In that case we have

$$y_{i+M}(x) = \exp\{-2\pi i (x - \rho_i)\}\; y_i(x). \tag{5.12}$$

With the aid of this relation the expression in the right-hand side of (5.11) can be converted to the form (5.1).

In general, we can choose the functions $\psi^{j\theta}$, for all $j \in \mathscr{J}_0$, in such a manner that the following relation holds

$$\psi^{j(\theta+2\pi)}(t) = \sum_{l \in \mathscr{J}_0} \mu_{jl} \psi^{l\theta}(t e^{-2\pi i}) \tag{5.13}$$

where the μ_{jl} are complex numbers, which are determined by the formal monodromy at the origin, i.e., by the relation

$$\hat{\psi}^j(t) = \sum_{l \in \mathscr{J}_0} \mu_{jl} \hat{\psi}^l(t e^{-2\pi i}).$$

From (5.13) we deduce the following generalization of (5.11)

$$y^{j\theta}(x) = \sum_{l \in \mathscr{J}_0} (I - e^{-2\pi i x} \mathscr{M})^{-1}_{jl} \sum_{i \in J_\theta} c_{li}\, y_i(x) \tag{5.14}$$

where $\mathscr{M}$ denotes the square matrix with entries μ_{jl} and J_θ is some finite set of integers. Suppose, for example, that (D) has the formal solutions

$$\hat{\psi}^1(t) = t^\rho \sum_{n=0}^{\infty} a_n t^n$$

and

$$\hat{\psi}^2(t) = \hat{\psi}^1(t) \log t + t^\rho \sum_{n=0}^{\infty} b_n t^n$$

and that $k^j > 0$ for all $j > 2$. In that case we can choose $\psi^{1\theta}$ and $\psi^{2\theta}$ in such a way that

$$\psi^{1(\theta+2\pi)}(t) = e^{2\pi i \rho} \psi^{1\theta}(t e^{-2\pi i})$$

and

$$\psi^{2(\theta+2\pi)}(t) = e^{2\pi i \rho}(\psi^{2\theta}(t e^{-2\pi i}) + 2\pi i \psi^{1\theta}(t e^{-2\pi i})).$$

Then we have

$$y^{1(\theta+2\pi)}(x)=e^{-2\pi i(x-\rho)}y^{1\theta}(x)$$

and

$$y^{2(\theta+2\pi)}(x)=e^{-2\pi i(x-\rho)}(y^{2\theta}(x)+2\pi i y^{1\theta}(x)).$$

Hence we deduce the relations

$$y^{1\theta}(x)=(1-e^{-2\pi i(x-\rho)})^{-1}\sum_{i\in J_\theta} c_{1i}y_i(x)$$

and

$$y^{2\theta}(x)=(1-e^{-2\pi i(x-\rho)})^{-1}\left\{\sum_{i\in J_\theta} c_{2i}y_i(x)+\frac{2\pi i e^{-2\pi i(x-\rho)}}{1-e^{-2\pi i(x-\rho)}}\sum_{i\in J_\theta} c_{1i}y_i(x)\right\}.$$

Note that, in this case, $y^{2\theta}$ (and hence the elements of the fundamental system $\tilde{Y}$) may have second order poles at the points $\rho+n$, $n\in\mathbb{N}\cup\{0\}$.

Returning to the original problem, let $\theta\in\mathbb{R}$ and let γ be the half line from O to ∞ with direction θ. Suppose that $\psi\in(\mathscr{A}_D^{\leqslant 0})_\theta$, i.e., ψ is a solution of (D) with moderate growth as $t\to 0$ in a small sector containing γ. Then ψ can be written in the form

$$\psi=\sum_{j\in\mathscr{J}_0} c^j\psi^{j\theta}+\phi,$$

where $c^j\in\mathbb{C}$ and $\phi\in(\mathscr{A}_D^{<0})_\theta$. Moreover,

$$\mathscr{P}_\gamma(\psi)=\sum_{j\in\mathscr{J}_0} c^j y^{j\theta}+\mathscr{P}_\gamma(\phi).$$

From Proposition 5.1 and (5.14) we derive the following theorem.

THEOREM 5.4. *Let $\theta\in\mathbb{R}$ and let γ be the half line from 0 to ∞ with direction θ. Suppose that $\psi\in(\mathscr{A}_D^{\leqslant 0})_\theta$. There exist complex numbers c^j, $j\in\mathscr{J}_0$ and c_i, $i\in J(\theta):=J_{\min\mathscr{J}}(\theta)$, such that*

$$\psi=\sum_{j\in\mathscr{J}_0} c^j\psi^{j\theta}+\sum_{i\in J(\theta)} c_i\psi_i.$$

For all $j\in\mathscr{J}_0$ there exist complex numbers c_{ji}, $i\in J_\theta$, such that

$$\psi^{j\theta}-\psi^{j(\theta+2\pi)}=\sum_{i\in J_\theta} c_{ji}\psi_i$$

and we have

$$\mathscr{P}_\gamma(\psi) = \sum_{j \in \mathscr{J}_0} c^j \sum_{l \in \mathscr{J}_0} (I - e^{-2\pi i x} \mathscr{M})^{-1}_{jl} \sum_{i \in J_\theta} c_{li} y_i + \sum_{i \in J(\theta)} c_i y_i. \qquad (5.15)$$

From Theorem 5.4 a connection formula for the fundamental systems Y and $\tilde{Y}$ can be derived. Let $i \in \mathbb{Z}$ and let $\tilde{y}_i$ be the solution of (Δ) defined in Definition 4.3, i.e.,

$$\tilde{y}_i = \sum_{v = M'_i - 1}^{N'_i} \mathscr{P}_{\gamma_i^v}(\psi_{i,v} - \psi_{i,v+1}).$$

For all $v \in \{M'_i - 1, ..., N'_i\}$, $\psi_{i,v} - \psi_{i,v+1} \in \mathscr{A}_D^{\leqslant 0}(I_{i,v} \cap I_{i,v+1})$. Let $\theta_v \in I_{i,v} \cap I_{i,v+1}$ denote the direction of γ_i^v. For all $j \in \mathscr{J}_0$ and all $h \in J(\theta_v)$ there exist complex numbers $c^j(i, v)$, $c_h(i, v)$ and $c_{jh}(i, v)$ such that

$$\psi_{i,v} - \psi_{i,v+1} = \sum_{j \in \mathscr{J}_0} c^j(i, v)\, \psi^{j\theta_v} + \sum_{h \in J(\theta_v)} c_h(i, v)\, \psi_h,$$

$$\psi^{j\theta_v} - \psi^{j(\theta_v + 2\pi)} = \sum_{i \in J\theta_v} c_{jh}(i, v)\, \psi_h$$

and

$$\mathscr{P}_{\gamma_i^v}(\psi_{i,v} - \psi_{i,v+1}) = \sum_{h \in J\theta_v} \sum_{j \in \mathscr{J}_0} c^j(i, v) \sum_{l \in \mathscr{J}_0} (I - e^{-2\pi i x} \mathscr{M})^{-1}_{jl}$$

$$\times c_{lh}(i, v)\, y_h + \sum_{h \in J(\theta_v)} c_h(i, v)\, y_h.$$

The numbers $c^j(i, v)$, $c_h(i, v)$ and $c_{jh}(i, v)$ are Stokes multipliers of the differential equation (D). As observed before, there is a great deal of arbitrariness in the choice of both fundamental systems and it is hard to give a general description of the sets J_θ and $J(\theta)$ that are involved in the connection between these systems. More precise results can be obtained for two particular systems, for example, if we take the solutions ψ_i and $\psi^{j\theta}$ $(j \in \mathscr{J}_0)$ to be specific "multi-sums" of $\hat{\psi}_i$ and $\hat{\psi}^j$, respectively. Suppose that ψ_i is defined as follows: ψ_i is the "multi-sum" of $\hat{\psi}_i$ in a direction $\alpha_i + \varepsilon$, $\varepsilon > 0$ and $\psi^{j\theta}$ is the multi-sum of $\hat{\psi}^j$ in the direction $\theta + \varepsilon$. If ε is sufficiently small we have

$$J_\theta \subset \{i \in \mathbb{Z}: \theta < \alpha_i \leqslant \theta + 2\pi\}.$$

The description of the set $J(\theta)$ for a given direction θ is slightly more complicated. A particularly simple situation arises when $k^1 > 0$. This is the case when, in (0.1), $a_{00} \neq 0$ and $a_{0l} = 0$ for all $l > 0$. Any solution y of the difference equation (Δ) which is analytic in some left half plane is an entire function, due to the relation

$$a_{00}\, y(x) = -\sum_{h=1}^{M} \sum_{l=0}^{m} a_{hl}(x-h)^l\, y(x-h).$$

The connection problem for the difference equation is completely determined by the Stokes phenomenon (at O) of (D): $\psi \in (\mathscr{A}_D^{\leqslant 0})_\theta$ implies $\psi \in (\mathscr{A}_D^{<0})_\theta$, hence there exist complex numbers (Stokes multipliers) c_i such that $\psi = \sum_{i \in J(\theta)} c_i \psi_i$ and $\mathscr{P}_\gamma(\psi) = \sum_{i \in J(\theta)} c_i y_i$.

EXAMPLE 5.5. Let

$$D = t^3 \partial^2 + t^3 \partial - 1.$$

The Newton polygon $N(D)$ has one positive slope, equal $\frac{3}{2}$. The differential equation (D) is essentially Airy's equation with the roles of O and ∞ interchanged. Let $\omega := e^{-2\pi i/3}$ and define

$$\psi_1(t) := -\omega^2 Ai\left(\frac{\omega^2}{t}\right), \qquad \psi_2(t) := \omega Ai\left(\frac{\omega}{t}\right), \qquad \psi_3(t) := Ai\left(\frac{1}{t}\right)$$

ψ_1, ψ_2 and ψ_3 are solutions of (D) with the following asymptotic properties:

$$\psi_1(t) \sim \hat{\psi}_1(t) := t^{1/4} e^{-2/3 t^{-3/2}} \hat{g}_1(t) \qquad \text{as} \quad t \to 0, \ -\frac{7\pi}{3} < \arg t < -\frac{\pi}{3},$$

$$\psi_2(t) \sim \hat{\psi}_2(t) := t^{1/4} e^{2/3 t^{-3/2}} \hat{g}_2(t) \qquad \text{as} \quad t \to 0, \ -\frac{5\pi}{3} < \arg t < \frac{\pi}{3},$$

and

$$\psi_3(t) \sim \hat{\psi}_3(t) := \hat{\psi}_1(t) \qquad \text{as} \quad t \to 0, \ |\arg t| < \pi,$$

where $\hat{g}_1, \hat{g}_2 \in \mathbb{C}[[t^{1/2}]]$. For all $i \in \mathbb{Z}$, let $\alpha_l = -2\pi + (2\pi/3)\, l$. Then we have (cf. Definition 2.1)

$$I_l = \left(-\frac{7\pi}{3} + \frac{2\pi}{3}\, l,\ -\frac{5\pi}{3} + \frac{2\pi}{3}\, l\right), \qquad \tilde{I}_l = \left(-3\pi + \frac{2\pi}{3}\, l,\ -\frac{7\pi}{3} + \frac{2\pi}{3}\, l\right).$$

Moreover, it is known that $\psi_3 + \psi_2 - \psi_1 = 0$.

The corresponding difference operator is

$$\Delta = (x-3)(x-2)\,\tau^{-3} - 1$$

and the difference equation (Δ) has a solution

$$y(x) := 9^{x/3}\Gamma\left(\frac{x}{3}\right)\Gamma\left(\frac{x+1}{3}\right),$$

analytic in a right half plane and admitting an asymptotic representation of the form

$$\hat{y}(x) = x^{2/3(x-1)}e^{-2x/3}\hat{h}(x),$$

as $x \to \infty$, $\arg x \in (-\pi/2, \pi/2)$, where $\hat{h} \in \mathbb{C}[[x^{-1}]]$. The solution y may be represented by the integral

$$y(x) = c\int_0^\infty Ai\left(\frac{1}{t}\right) t^{-x-1}\,dt = c\int_0^\infty Ai(t)\,t^{x-1}\,dt$$

where c is a complex number which can be determined by comparing the asymptotic representations of y and the Mellin transform of Airy's function. Furthermore, (Δ) has a solution

$$\tilde{y}(x) := -4\pi^2 e^{(\pi/3)\,i(2x+1)}9^{x/3}\Gamma\left(1-\frac{x}{3}\right)^{-1}\Gamma\left(\frac{2-x}{3}\right)^{-1},$$

analytic in a left half plane and admitting the asymptotic representation $\hat{y}(x)$ as $x \to \infty$, $\arg x \in (\pi/2, 3\pi/2)$. Now, let

$$y_1(x) := \int_0^{\omega^2\cdot\infty} \psi_1(t)\,t^{-x-1}\,dt,$$

$$y_2(x) := \int_0^{\omega\cdot\infty} \psi_2(t)\,t^{-x-1}\,dt,$$

$$y_3(x) := \int_0^{\infty} \psi_3(t)\,t^{-x-1}\,dt.$$

The following relations are readily verified

$$y_1(x) = -e^{(4/3)\pi i(x-1)}y_3(x), \qquad y_2(x) = e^{(2/3)\pi i(x-1)}y_3(x), \qquad y(x) = cy_3(x). \tag{5.16}$$

The functions y_1, y_2 and y_3 constitute a fundamental system of solutions of (Δ) of the type defined in Section 3.

Next, we are going to apply the theory of Section 4, in order to obtain a solution $\tilde{y}_3$ of (Δ), represented asymptotically by $\hat{y}_3 := (1/c)\,\hat{y}$ as $x \to \infty$, $\arg x \in (\pi/2, 3\pi/2)$. We take $M'_3 = 0$, $N_3 = 1$, $N'_3 = 2$ and define

$$I_{3,0} = \left(-\frac{5\pi}{3}, -\frac{2\pi}{3} + \varepsilon\right), \qquad I_{3,1} = \tilde{I}_3, \qquad I_{3,2} = \left(-\frac{\pi}{3} - \varepsilon, \frac{\pi}{3}\right)$$

where $\varepsilon \in (0, \pi/6)$, and

$$\psi_{3,0} = \psi_1, \qquad \psi_{3,1} = \psi_{3,2} = \psi_3.$$

Due to the fact that $\psi_{3,0} - \psi_{3,1} = \psi_1 - \psi_3 = \psi_2$, it is easily seen that the conditions of Lemma 4.1 are satisfied, with $\tilde{\psi}_{3,\nu} = \psi_{3,\nu}$ for $\nu = 0$ and $\nu = 2$. Furthermore we take γ_3^{-1}, γ_3^0 and γ_3^2 to be the half lines from O to ∞ with directions $-(2\pi/3)$, $-(2\pi/3)$ and 0, respectively. In accordance with Definition 4.3 we set

$$\tilde{y}_3 = -\mathscr{P}_{\gamma_3^{-1}}(\psi_1) + \mathscr{P}_{\gamma_3^0}(\psi_1 - \psi_3) + \mathscr{P}_{\gamma_3^2}(\psi_3),$$

i.e., $\tilde{y}_3 = y_3 + y_2 - y_1$. By Theorem 4.5, $\tilde{y}_3(x) \sim \hat{y}_3(x)$ as $x \to \infty$, $\arg x \in (\pi/2, 3\pi/2)$. Hence it can be seen that $\tilde{y} = c\tilde{y}_3$. With (5.16) it follows that

$$\frac{\tilde{y}_3}{y_3} = 1 + e^{(2/3)\,\pi i(x-1)} + e^{(4/3)\,\pi i(x-1)}$$

and the right-hand side is equal to $\tilde{y}/y$ as was to be expected.

EXAMPLE 5.6. Let

$$D = \alpha - \partial + t\partial^2 - t^3\partial^3$$

where $\alpha \in \mathbb{C}^*$. D is regular at ∞. At O, the differential equation (D) has three formal solutions $\hat{\psi}^1$, $\hat{\psi}^2$, and $\hat{\psi}^3$, of the form

$$\hat{\psi}^1(t) = t^\alpha \sum_{n=0}^{\infty} a_n t^n$$

$$\hat{\psi}^2(t) = e^{-t^{-1}} t^{-\alpha} \sum_{n=0}^{\infty} b_n t^{n+2}$$

and

$$\hat{\psi}^3(t) = e^{-(1/2)\,t^{-2} + t^{-1}} \sum_{n=0}^{\infty} c_n t^{n+3}.$$

For $i=1,2$ and every $l\in\mathbb{Z}$, (D) has a unique solution ψ_l^i, analytic in a reduced neighbourhood of the origin, with the property that

$$\psi_l^i(t)\sim\hat{\psi}^i(t)\qquad \text{as}\quad t\to 0,\quad \arg t\in\left(-\frac{5\pi}{4},\frac{\pi}{4}\right)+l\pi$$

and a solution ψ_l^3 with the property that

$$\psi_l^3(t)\sim\hat{\psi}^3(t)\qquad \text{as}\quad t\to 0,\quad \arg t\in\left(-\frac{3\pi}{4},\frac{3\pi}{4}\right)+l\pi.$$

Obviously,

$$\psi_{l+2}^1=e^{2\pi i\alpha}\psi_l^1,\qquad \psi_{l+2}^2=e^{-2\pi i\alpha}\psi_l^2\qquad \text{and}\qquad \psi_{l+2}^3=\psi_l^3. \tag{5.17}$$

The Stokes phenomenon at O is described by the following relations:

$$\begin{aligned}\psi_l^1-\psi_{l+1}^1&=s_{12}^l\psi_l^2+s_{13}^l\psi_l^3,\\ \psi_l^2-\psi_{l+1}^2&=s_{21}^l\psi_l^1+s_{23}^l\psi_l^3,\\ \psi_l^3-\psi_{l+1}^3&=s_{31}^l\psi_{l+1}^1+s_{32}^l\psi_{l+1}^2,\end{aligned} \tag{5.18}$$

where $s_{12}^{2l+1}=0$ and $s_{21}^{2l}=0$ for all $l\in\mathbb{Z}$. For all $j\in\mathbb{Z}$ we define

$$\alpha_{3j}=\alpha_{3j+1}=2j\pi,\qquad \alpha_{3j+2}=(2j+1)\,\pi,$$
$$k_{3j}=1\qquad \text{and}\qquad k_i=2\qquad \text{for all}\quad i\notin 3\mathbb{Z}.$$

Consequently,

$$I_{3j}=\left(2j\pi-\frac{\pi}{2},2j\pi+\frac{\pi}{2}\right),$$

$$I_{3j+1}=\left(2j\pi-\frac{\pi}{4},2j\pi+\frac{\pi}{4}\right),$$

$$I_{3j+2}=\left(2j\pi+\frac{3\pi}{4},2j\pi+\frac{5\pi}{4}\right)$$

and

$$\tilde{I}_{3j}=\left(2j\pi-\frac{3\pi}{2},2j\pi-\frac{\pi}{2}\right),$$

$$\tilde{I}_{3j+1}=\left(2j\pi-\frac{3\pi}{4},2j\pi-\frac{\pi}{4}\right),$$

$$\tilde{I}_{3j+2}=\left(2j\pi+\frac{\pi}{4},2j\pi+\frac{3\pi}{4}\right).$$

We define the functions ψ_i, $i \in \mathbb{Z}$, in the following way,

$$\psi_{3j} = \psi^2_{2j}, \qquad \psi_{3j+1} = \psi^3_{2j} \qquad \text{and} \qquad \psi_{3j+2} = \psi^3_{2j+1}.$$

The corresponding difference equation (Δ) reads

$$(\alpha - x)\, y(x) + (x-1)^2\, y(x-1) - (x-3)^3\, y(x-3) = 0.$$

It has formal solutions of the form

$$\begin{aligned}
\hat{y}_{3j}(x) &= e^{-2j\pi i(x+\alpha) - x} x^{x+\alpha-5/2} \hat{f}_0(x)\\
\hat{y}_{3j+1}(x) &= e^{-2j\pi ix - (1/2)(x - x^{1/2})} x^{x/2-2} \hat{f}_1(x)\\
\hat{y}_{3j+2}(x) &= e^{-(2j+1)\pi ix - (1/2)(x + x^{1/2})} x^{x/2-2} \hat{f}_2(x)
\end{aligned}$$

where $j \in \mathbb{Z}$, $\hat{f}_0 \in \mathbb{C}[\![x^{-1}]\!]$, $\hat{f}_1 \in \mathbb{C}[\![x^{-1/2}]\!]$ and $\hat{f}_2(x) = \hat{f}_1(xe^{-2\pi i})$.

For all $\theta \in \mathbb{R}$ let γ_θ denote the half line from O to ∞ with direction θ. Then we have

$$y_{3j} = \mathscr{P}_{\gamma_{2j\pi}}(\psi^2_{2j}), \qquad y_{3j+1} = \mathscr{P}_{\gamma_{2j\pi}}(\psi^3_{2j}), \qquad y_{3j+2} = \mathscr{P}_{\gamma_{(2j+1)\pi}}(\psi^3_{2j+1}).$$

From (5.17) we deduce the relations (cf. Remark 5.3)

$$y_{3j+3} = e^{-2\pi i(x+\alpha)} y_{3j}, \quad y_{3j+4} = e^{-2\pi ix} y_{3j+1}, \quad \text{and} \quad y_{3j+5} = e^{-2\pi ix} y_{3j+2}. \tag{5.19}$$

For all $i \in \mathbb{Z}$ we define the functions $\tilde{y}_i$ by

$$\begin{aligned}
\tilde{y}_{3j} &= \mathscr{P}_{\gamma_{2j\pi}}(\psi^2_{2j}) + \mathscr{P}_{\gamma_{(2j-1)\pi}}(\psi^2_{2j-1} - \psi^2_{2j}) - \mathscr{P}_{\gamma_{2(j-1)\pi}}(\psi^2_{2j-1})\\
\tilde{y}_{3j+1} &= \mathscr{P}_{\gamma_{2j\pi}}(\psi^3_{2j}) + \mathscr{P}_{\gamma_{(2j-1/2)\pi}}(\psi^3_{2j-1} - \psi^3_{2j}) - \mathscr{P}_{\gamma_{(2j-1)\pi}}(\psi^3_{2j-1})\\
\tilde{y}_{3j+2} &= \mathscr{P}_{\gamma_{(2j+1)\pi}}(\psi^3_{2j+1}) + \mathscr{P}_{\gamma_{(2j+1/2)\pi}}(\psi^3_{2j} - \psi^3_{2j+1}) - \mathscr{P}_{\gamma_{2j\pi}}(\psi^3_{2j}).
\end{aligned}$$

(These expressions are in agreement with Definition 5.11, provided we take $M'_i = 0$ for all i, $N'_{3j} = 3$, $N'_{3j+1} = N'_{3j+2} = 2$, $\psi_{3j,0} = \psi_{3j,1} = \psi^2_{2j-1}$ and $\psi_{3j,2} = \psi_{3j,3} = \psi^2_{2j}$, $\psi_{3j+1,0} = \psi^3_{2j-1}$ and $\psi_{3j+1,1} = \psi_{3j+1,2} = \psi^3_{2j}$, $\psi_{3j+2,0} = \psi_{3j+2,1} = \psi^3_{2j}$ and $\psi_{3j+2,2} = \psi^3_{2j+1}$.) From (5.17) and (5.18) we derive the identities (cf. Proposition 5.1)

$$\mathscr{P}_{\gamma_{2j\pi}}(\psi^2_{2j+1}) = e^{-2\pi i(x+\alpha)} \mathscr{P}_{\gamma_{2(j-1)\pi}}(\psi^2_{2j-1}) = y_{3j} - s^{2j}_{23} y_{3j+1}, \qquad j \in \mathbb{Z} \tag{5.20}$$

and (cf. Lemma 5.2)

$$\begin{aligned}
P_{\gamma_{l\pi}}(\psi^1_l) - \mathscr{P}_{\gamma_{(l+2)\pi}}(\psi^1_{l+2}) &= s^l_{12} \mathscr{P}_{\gamma_{l\pi}}(\psi^2_l) + s^l_{13} \mathscr{P}_{\gamma_{l\pi}}(\psi^3_l)\\
&\quad + s^{l+1}_{12} \mathscr{P}_{\gamma_{(l+1)\pi}}(\psi^2_{l+1}) + s^{l+1}_{13} \mathscr{P}_{\gamma_{(l+1)\pi}}(\psi^3_{l+1}), \qquad l \in \mathbb{Z}.
\end{aligned}$$

From (5.17) it also follows that, for all $l \in \mathbb{Z}$,

$$\mathscr{P}_{\gamma(l+2)\pi}(\psi^1_{l+2}) = e^{-2\pi i(x-\alpha)} \mathscr{P}_{\gamma l\pi}(\psi^1_l). \tag{5.21}$$

Hence we conclude that

$$\mathscr{P}_{\gamma(2j-1)\pi}(\psi^1_{2j-1}) = (1-e^{-2\pi i(x-\alpha)})^{-1} \{s_{13}^{2j-1} e^{2\pi i x} y_{3j+2} + s_{12}^{2j} y_{3j} + s_{13}^{2j} y_{3j+1}\} \tag{5.22}$$

and

$$\mathscr{P}_{\gamma 2j\pi}(\psi^1_{2j}) = (1-e^{-2\pi i(x-\alpha)})^{-1} \{s_{12}^{2j} y_{3j} + s_{13}^{2j} y_{3j+1} + s_{13}^{2j+1} y_{3j+2}\}. \tag{5.23}$$

Using (5.17)–(5.23) we obtain the following connection formula for the fundamental systems $\{\tilde{y}_{3j}, \tilde{y}_{3j+1}, \tilde{y}_{3j+2}\}$ and $\{y_{3j}, y_{3j+1}, y_{3j+2}\}$ of (Δ):

$$\begin{aligned}
\tilde{y}_{3j} = &\left(1 - e^{2\pi i(x+\alpha)} + \frac{s_{21}^{2j-1} s_{12}^{2j}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j} \\
&+ \left(s_{23}^{2j} e^{2\pi i(x+\alpha)} + \frac{s_{21}^{2j-1} s_{13}^{2j}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j+1} \\
&+ \left(s_{23}^{2j-1} e^{2\pi i x} + \frac{s_{21}^{2j-1} s_{13}^{2j-1} e^{2\pi i x}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j+2} \\
\tilde{y}_{3j+1} = &\left(s_{32}^{2j-1} + \frac{s_{31}^{2j-1} s_{12}^{2j}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j} + \left(1 + \frac{s_{31}^{2j-1} s_{13}^{2j}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j+1} \\
&+ \left(-e^{2\pi i x} + \frac{s_{31}^{2j-1} s_{13}^{2j+1}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j+2} \\
\tilde{y}_{3j+2} = &\left(s_{32}^{2j} + \frac{s_{31}^{2j} s_{12}^{2j} e^{-2\pi i(x-\alpha)}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j} \\
&+ \left(-1 - s_{32}^{2j} s_{23}^{2j} + \frac{s_{31}^{2j} s_{13}^{2j} e^{-2\pi i(x-\alpha)}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j+1} \\
&+ \left(1 + \frac{s_{31}^{2j} s_{13}^{2j-1} e^{2\pi i\alpha}}{1-e^{-2\pi i(x-\alpha)}}\right) y_{3j+2}.
\end{aligned}$$

From (5.17) and (5.18) we deduce the relations

$$s_{13}^l s_{31}^{l-1} = e^{2\pi i\alpha} - 1 \qquad \text{and} \qquad s_{23}^l s_{32}^{l-1} = e^{-2\pi i\alpha} - 1 \qquad \text{for all} \quad l \in \mathbb{Z},$$

$$s_{13}^{2j} s_{32}^{2j-1} = s_{12}^{2j}, \qquad s_{23}^{2j} s_{31}^{2j-1} = e^{-2\pi i\alpha} s_{21}^{2j-1},$$

$$s_{31}^{2j} = -s_{31}^{2j-1} = e^{-2\pi i\alpha} s_{31}^{2j-2}$$

and

$$s_{32}^{2j+1} = -s_{32}^{2j} = e^{2\pi i\alpha} s_{32}^{2j-1} \qquad \text{for all} \quad j \in \mathbb{Z}.$$

Hence it follows that

$$\tilde{y}_{3j} = \frac{1-e^{-2\pi i x}}{1-e^{-2\pi i(x-\alpha)}} \{e^{2\pi i\alpha}(1-e^{2\pi i x})\, y_{3j} + s_{23}^{2j} e^{2\pi i(x+\alpha)}(y_{3j+1} - y_{3j+2})\}$$

$$\tilde{y}_{3j+1} = \frac{1-e^{-2\pi i x}}{1-e^{-2\pi i(x-\alpha)}} \{-s_{32}^{2j} y_{3j} + e^{2\pi i\alpha} y_{3j+1} - e^{2\pi i x} y_{3j+2}\}$$

$$\tilde{y}_{3j+2} = \frac{1-e^{-2\pi i x}}{1-e^{-2\pi i(x-\alpha)}} \{s_{32}^{2j} y_{3j} - e^{2\pi i\alpha} y_{3j+1} + e^{2\pi i\alpha} y_{3j+2}\}.$$

These expressions show that $\tilde{y}_{3j}$, $\tilde{y}_{3j+1}$ and $\tilde{y}_{3j+2}$ behave approximately as y_{3j}, $-s_{32}^{2j} e^{2\pi i\alpha} y_{3j}$ and $s_{32}^{2j} e^{2\pi i\alpha} y_{3j}$, respectively, as $x \to \infty$, $0 < \arg x < \pi/2$.

References

1. W. Balser, B. L. J. Braaksma, J.-P. Ramis, and Y. Sibuya, Multisummability of formal power series solutions of linear ordinary differential equations, *Asymptotic Anal.* **5** (1991), 27–45.
2. A. Barkatou, "Contribution á l'étude des équations différentielles et aux différences dans le champ complexe," thesis, 1988.
3. B. L. J. Braaksma, G. K. Immink, and Y. Sibuya, The Stokes phenomenon in exact asymptotics, preprint.
4. N. G. de Bruijn, "Asymptotic Methods in Analysis," North-Holland, Amsterdam, 1961.
5. A. Duval, Equations aux différences algébriques: solutions méromorphes dans $\mathbb{C} - \mathbb{R}^-$. Système fondamental de solutions holomorphes dans un demi-plan, *in* "Equations différentielles et systèmes de Pfaff dans le champ complexe-II," Lecture Notes in Mathematics, Vol. 1015, pp. 102–135, Springer-Verlag, Berlin/New York, 1983.
6. A. Duval, Lemmes de Hensel et factorisation formelle pour les opérateurs aux différences, *Funkcial. Ekvac.* **26** (1983), 349–368.
7. G. K. Immink, On the relation between global properties of linear difference and differential equations with polynomial coefficients, I, *J. Differential Equations* **113**, No. 1 (1994), 201–234.
8. G. K. Immink, On the relation between global properties of linear difference and differential equations with polynomial coefficients, III, preprint, 1994.
9. N. E. Nörlund, "Vorlesungen über Differenzenrechnung," Chelsea, New York, 1954.
10. J. P. Ramis, Étude des solutions méromorphes des équations aux différences linéaires algébriques, manuscript.
11. E. C. Titchmarsh, "The Theory of Functions," 2nd ed., Oxford Univ. Press, Oxford, 1939.